2018 SchweserNotes™ Part II

FRM® Exam Prep

Market Risk Measurement and Management

eBook 1

KAPLAN SCHWESER

Getting Started

FRM®

FRM® Exam Part II

Welcome

As the VP of Advanced Designations at Kaplan Schweser, I am pleased to have the opportunity to help you prepare for the 2018 FRM® Exam. Getting an early start on your study program is important for you to sufficiently **prepare**, **practice**, and **perform** on exam day. Proper planning will allow you to set aside enough time to master the learning objectives in the Part II curriculum.

Now that you've received your SchweserNotes™, here's how to get started:

Step 1: Access Your Online Tools

Visit www.schweser.com/frm and log in to your online account using the button located in the top navigation bar. After logging in, select the appropriate part and proceed to the dashboard where you can access your online products.

Step 2: Create a Study Plan

Create a study plan with the Schweser Study Calendar (located on the Schweser dashboard). Then view the Candidate Resource Library on-demand videos for an introduction to core concepts.

Step 3: Prepare and Practice

Read your SchweserNotes™

Our clear, concise study notes will help you **prepare** for the exam. At the end of each reading, you can answer the Concept Checker questions for better understanding of the curriculum.

Attend a Weekly Class

Attend our Live Online Weekly Class or review the on-demand archives as often as you like. Our expert faculty will guide you through the FRM curriculum with a structured approach to help you **prepare** for the exam. (See our instruction packages to the right. Visit www.schweser.com/frm to order.)

Practice with SchweserPro™ QBank

Maximize your retention of important concepts and **practice** answering exam-style questions in the SchweserPro™ QBank and taking several Practice Exams. Use Schweser's QuickSheet for continuous review on the go. (Visit www.schweser.com/frm to order.)

Step 4: Final Review

A few weeks before the exam, make use of our Online Review Workshop Package. Review key curriculum concepts in every topic, **perform** by working through demonstration problems, and **practice** your exam techniques with our 8-hour live Online Review Workshop. Use Schweser's Secret Sauce® for convenient study on the go.

Step 5: Perform

As part of our Online Review Workshop Package, take a Schweser Mock Exam to ensure you are ready to **perform** on the actual FRM Exam. Put your skills and knowledge to the test and gain confidence before the exam..

Again, thank you for trusting Kaplan Schweser with your FRM Exam preparation!

Sincerely,

Derek Burkett

Derek Burkett, CFA, FRM, CAIA
VP, Advanced Designations, Kaplan Schweser

The Kaplan Way

Prepare

Acquire new knowledge through demonstration and examples.

Practice

Apply new knowledge through simulation and practice.

Perform

Evaluate mastery of new knowledge and identify achieved outcomes.

FRM® Instruction Packages:

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Remember to join our Live Online Weekly Class.
Register online today at www.schweser.com/frm.

*Dates, times, and instructors subject to change

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FRM Part II Book 1: Market Risk Measurement and Management

FRM 2018 PART II BOOK 1: MARKET RISK MEASUREMENT AND MANAGEMENT

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Welcome to the 2018 SchweserNotes

Thank you for trusting Kaplan Schweser to help you reach your career and educational goals. We are very pleased to be able to help you prepare for the FRM Part II exam. In this introduction, I want to explain the resources included with the SchweserNotes, suggest how you can best use Schweser materials to prepare for the exam, and direct you toward other educational resources you will find helpful as you study for the exam.

Besides the SchweserNotes themselves, there are many online educational resources available at Schweser.com. Just log in using the individual username and password you received when you purchased the SchweserNotes.

SchweserNotes™

The SchweserNotes consist of four volumes that include complete coverage of all FRM assigned topics and learning objectives (LOs), Concept Checkers (multiple-choice questions for every topic), and Self-Test questions to help you master the material and check your retention of key concepts.

Online Practice Questions

To retain what you learn, it is important that you quiz yourself often. We offer an online version of the SchweserPro™ QBank, which contains hundreds of Part II practice questions and explanations. Quizzes are available for each topic or across multiple topics. Build your own exams by specifying the topics and the number of questions.

Practice Exams

Schweser offers two full 4-hour practice exams. These exams are important tools for gaining the speed and skills you will need to pass the exam. The Practice Exams book contains answers with full explanations for self-grading and evaluation.

Schweser Study Calendar

Use your Online Access to tell us when you will start and what days of the week you can study. The online Schweser Study Calendar will create a study plan just for you, breaking the curriculum into daily and weekly tasks to keep you on track and help you monitor your study progress.

The FRM Part II exam is a formidable challenge (covering 79 assigned readings and almost 500 learning objectives), and you must devote considerable time and effort to be properly prepared. There are no shortcuts! You must learn the material, know the terminology and techniques, understand the concepts, and be able to answer 80 multiple choice questions quickly and (at least 70%) correctly. A good estimate of the study time required on average is 250 hours, but some candidates will need more or less time, depending on their individual backgrounds and experience.

To help you really master this material and be well-prepared for the FRM exam, we offer several other educational resources, including:

Online Weekly Class

Our Online Weekly Class is offered each week, beginning in February for the May exam and August for the November exam. This online class brings the personal attention of a classroom into your home or office with 30 hours of real-time instruction, led by David McMeekin, CFA, CAIA, FRM. The class offers in-depth coverage of difficult concepts, instant feedback during lecture and Q&A sessions, and discussion of sample exam questions. Archived classes are available for viewing at any time throughout the season. Candidates enrolled in the Online Weekly Class also have full access to supplemental on-demand video instruction in the Candidate Resource Library and an e-mail address link for sending questions to the instructor at any time.

Late-Season Review

Late-season review and exam practice can make all the difference. Our Review Package helps you evaluate your exam readiness with products specifically designed for late-season studying. This Review Package includes the Online Review Workshop (8-hour live and archived online review of essential curriculum topics), the Schweser Mock Exam (one 4-hour exam), and Schweser's Secret Sauce® (concise summary of the FRM curriculum).

Part II Exam Weights

In preparing for the exam, pay attention to the weights assigned to each knowledge domain within the curriculum. The Part II exam weights are as follows:

Book	*Knowledge Domains*	*Exam Weight*	*Exam Questions*
1	Market Risk Measurement and Management	25%	20
2	Credit Risk Measurement and Management	25%	20
3	Operational and Integrated Risk Management	25%	20
4	Risk Management and Investment Management	15%	12
4	Current Issues in Financial Markets	10%	8

How to Succeed

There are no shortcuts to studying for this exam. Expect the Global Association of Risk Professionals (GARP) to test you in a way that will reveal how well you know the Part II curriculum. You should begin studying early and stick to your study plan. You should first read the SchweserNotes and complete the Concept Checkers for each topic. At the end of each book, you should answer the provided Self-Test questions to understand how concepts may be tested on the exam. You should finish the overall curriculum at least two weeks before the FRM exam. This will allow sufficient time for Practice Exams and further review of those topics you have not yet mastered.

I would like to take this opportunity to thank the content developers, editors, and graphic designers who worked countless hours to create the 2018 FRM SchweserNotes. I would especially like to thank Derek Burkett, CFA, FRM, CAIA; Adam Stueber, CAIA; Craig Prochaska, CFA; Kent Westlund, CFA, FRM; Kurt Schuldes, CFA, CAIA; Tim Greive, CFA, CAIA; Jeff Bahr, Andy Bauer, Allie Bottcher, Katherine Bourgeois, Alyssa Brunner, Lindsey Casto, Laura Goetzinger, Ryan Henry, Hannah Kelley, Alissa Knop, Genevieve Kretschmer, Gretchen Panzer, Jessica Pearse, Ashley Sinclair, Ben Strong, and Debbie White for their contributions.

Best regards,

Eric Smith

Eric Smith, CFA, FRM
Content Manager
Kaplan Schweser

Reading Assignments and Learning Objectives

The following material is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by the Global Association of Risk Professionals.

Reading Assignments

Kevin Dowd, *Measuring Market Risk, 2nd Edition* (West Sussex, U.K.: John Wiley & Sons, 2005).

Philippe Jorion, *Value-at-Risk: The New Benchmark for Managing Financial Risk, 3rd Edition* (New York, NY: McGraw Hill, 2007).

Gunter Meissner, *Correlation Risk Modeling and Management* (New York, NY: John Wiley & Sons, 2014).

Bruce Tuckman and Angel Serrat, *Fixed Income Securities, 3rd Edition* (Hoboken, NJ: John Wiley & Sons, 2011).

Learning Objectives

1. Estimating Market Risk Measures: An Introduction and Overview

After completing this reading, you should be able to:
1. Estimate VaR using a historical simulation approach. (page 2)
2. Estimate VaR using a parametric approach for both normal and lognormal return distributions. (page 4)
3. Estimate the expected shortfall given P/L or return data. (page 6)
4. Define coherent risk measures. (page 6)
5. Estimate risk measures by estimating quantiles. (page 6)
6. Evaluate estimators of risk measures by estimating their standard errors. (page 7)
7. Interpret QQ plots to identify the characteristics of a distribution. (page 9)

2. Non-parametric Approaches

After completing this reading, you should be able to:
1. Apply the bootstrap historical simulation approach to estimate coherent risk measures. (page 15)
2. Describe historical simulation using non-parametric density estimation. (page 16)
3. Compare and contrast the age-weighted, the volatility-weighted, the correlation-weighted, and the filtered historical simulation approaches. (page 17)
4. Identify advantages and disadvantages of non-parametric estimation methods. (page 19)

3. Backtesting VaR

After completing this reading, you should be able to:
1. Define backtesting and exceptions and explain the importance of backtesting VaR models. (page 25)
2. Explain the significant difficulties in backtesting a VaR model. (page 26)
3. Verify a model based on exceptions or failure rates. (page 26)
4. Define and identify Type I and Type II errors. (page 28)
5. Explain the need to consider conditional coverage in the backtesting framework. (page 32)
6. Describe the Basel rules for backtesting. (page 33)

4. VaR Mapping

After completing this reading, you should be able to:
1. Explain the principles underlying VaR mapping, and describe the mapping process. (page 38)
2. Explain how the mapping process captures general and specific risks. (page 39)
3. Differentiate among the three methods of mapping portfolios of fixed income securities. (page 41)
4. Summarize how to map a fixed income portfolio into positions of standard instruments. (page 41)
5. Describe how mapping of risk factors can support stress testing. (page 44)
6. Explain how VaR can be used as a performance benchmark. (page 45)
7. Describe the method of mapping forwards, forward rate agreements, interest rate swaps, and options. (page 48)

5. **Messages from the Academic Literature on Risk Measurement for the Trading Book**

After completing this reading, you should be able to:
1. Explain the following lessons on VaR implementation: time horizon over which VaR is estimated, the recognition of time varying volatility in VaR risk factors, and VaR backtesting. (page 56)
2. Describe exogenous and endogenous liquidity risk and explain how they might be integrated into VaR models. (page 57)
3. Compare VaR, expected shortfall, and other relevant risk measures. (page 57)
4. Compare unified and compartmentalized risk measurement. (page 58)
5. Compare the results of research on "top-down" and "bottom-up" risk aggregation methods. (page 59)
6. Describe the relationship between leverage, market value of asset, and VaR within an active balance sheet management framework. (page 60)

6. **Some Correlation Basics: Properties, Motivation, Terminology**

After completing this reading, you should be able to:
1. Describe financial correlation risk and the areas in which it appears in finance. (page 64)
2. Explain how correlation contributed to the global financial crisis of 2007 to 2009. (page 74)
3. Describe the structure, uses, and payoffs of a correlation swap. (page 70)
4. Estimate the impact of different correlations between assets in the trading book on the VaR capital charge. (page 71)
5. Explain the role of correlation risk in market risk and credit risk. (page 76)
6. Relate correlation risk to systemic and concentration risk. (page 76)

7. **Empirical Properties of Correlation: How Do Correlations Behave in the Real World?**

After completing this reading, you should be able to:
1. Describe how equity correlations and correlation volatilities behave throughout various economic states. (page 88)
2. Calculate a mean reversion rate using standard regression and calculate the corresponding autocorrelation. (page 89)
3. Identify the best-fit distribution for equity, bond, and default correlations. (page 92)

8. **Statistical Correlation Models—Can We Apply Them to Finance?**

After completing this reading, you should be able to:
1. Evaluate the limitations of financial modeling with respect to the model itself, calibration of the model, and the model's output. (page 98)
2. Assess the Pearson correlation approach, Spearman's rank correlation, and Kendall's τ, and evaluate their limitations and usefulness in finance. (page 100)

9. **Financial Correlation Modeling—Bottom-Up Approaches**

After completing this reading, you should be able to:
1. Explain the purpose of copula functions and the translation of the copula equation. (page 111)
2. Describe the Gaussian copula and explain how to use it to derive the joint probability of default of two assets. (page 112)
3. Summarize the process of finding the default time of an asset correlated to all other assets in a portfolio using the Gaussian copula. (page 115)

10. Empirical Approaches to Risk Metrics and Hedging

After completing this reading, you should be able to:
1. Explain the drawbacks to using a DV01-neutral hedge for a bond position. (page 121)
2. Describe a regression hedge and explain how it can improve a standard DV01-neutral hedge. (page 122)
3. Calculate the regression hedge adjustment factor, beta. (page 123)
4. Calculate the face value of an offsetting position needed to carry out a regression hedge. (page 123)
5. Calculate the face value of multiple offsetting swap positions needed to carry out a two-variable regression hedge. (page 124)
6. Compare and contrast level and change regressions. (page 125)
7. Describe principal component analysis and explain how it is applied to constructing a hedging portfolio. (page 125)

11. The Science of Term Structure Models

After completing this reading, you should be able to:
1. Calculate the expected discounted value of a zero-coupon security using a binomial tree. (page 132)
2. Construct and apply an arbitrage argument to price a call option on a zero-coupon security using replicating portfolios. (page 132)
3. Define risk-neutral pricing and apply it to option pricing. (page 135)
4. Distinguish between true and risk-neutral probabilities, and apply this difference to interest rate drift. (page 135)
5. Explain how the principles of arbitrage pricing of derivatives on fixed income securities can be extended over multiple periods. (page 136)
6. Define option-adjusted spread (OAS) and apply it to security pricing. (page 141)
7. Describe the rationale behind the use of recombining trees in option pricing. (page 138)
8. Calculate the value of a constant maturity Treasury swap, given an interest rate tree and the risk-neutral probabilities. (page 139)
9. Evaluate the advantages and disadvantages of reducing the size of the time steps on the pricing of derivatives on fixed income securities. (page 142)
10. Evaluate the appropriateness of the Black-Scholes-Merton model when valuing derivatives on fixed income securities. (page 142)
11. Describe the impact of embedded options on the value of fixed income securities. (page 143)

12. The Evolution of Short Rates and the Shape of the Term Structure

After completing this reading, you should be able to:
1. Explain the role of interest rate expectations in determining the shape of the term structure. (page 149)
2. Apply a risk-neutral interest rate tree to assess the effect of volatility on the shape of the term structure. (page 151)
3. Estimate the convexity effect using Jensen's inequality. (page 153)
4. Evaluate the impact of changes in maturity, yield, and volatility on the convexity of a security. (page 153)
5. Calculate the price and return of a zero coupon bond incorporating a risk premium. (page 157)

13. The Art of Term Structure Models: Drift

After completing this reading, you should be able to:

1. Construct and describe the effectiveness of a short term interest rate tree assuming normally distributed rates, both with and without drift. (page 164)
2. Calculate the short-term rate change and standard deviation of the rate change using a model with normally distributed rates and no drift. (page 165)
3. Describe methods for addressing the possibility of negative short-term rates in term structure models. (page 166)
4. Construct a short-term rate tree under the Ho-Lee Model with time-dependent drift. (page 168)
5. Describe uses and benefits of the arbitrage-free models and assess the issue of fitting models to market prices. (page 168)
6. Describe the process of constructing a simple and recombining tree for a short-term rate under the Vasicek Model with mean reversion. (page 169)
7. Calculate the Vasicek Model rate change, standard deviation of the rate change, expected rate in T years, and half life. (page 172)
8. Describe the effectiveness of the Vasicek Model. (page 173)

14. The Art of Term Structure Models: Volatility and Distribution

After completing this reading, you should be able to:

1. Describe the short-term rate process under a model with time-dependent volatility. (page 179)
2. Calculate the short-term rate change and determine the behavior of the standard deviation of the rate change using a model with time dependent volatility. (page 179)
3. Assess the efficacy of time-dependent volatility models. (page 180)
4. Describe the short-term rate process under the Cox-Ingersoll-Ross (CIR) and lognormal models. (page 181)
5. Calculate the short-term rate change and describe the basis point volatility using the CIR and lognormal models. (page 181)
6. Describe lognormal models with deterministic drift and mean reversion. (page 183)

15. Volatility Smiles

After completing this reading, you should be able to:

1. Define volatility smile and volatility skew. (page 190)
2. Explain the implications of put-call parity on the implied volatility of call and put options. (page 189)
3. Compare the shape of the volatility smile (or skew) to the shape of the implied distribution of the underlying asset price and to the pricing of options on the underlying asset. (page 190)
4. Describe characteristics of foreign exchange rate distributions and their implications on option prices and implied volatility. (page 191)
5. Describe the volatility smile for equity options and foreign currency options and provide possible explanations for its shape. (page 191)
6. Describe alternative ways of characterizing the volatility smile. (page 192)
7. Describe volatility term structures and volatility surfaces and how they may be used to price options. (page 193)
8. Explain the impact of the volatility smile on the calculation of the "Greeks." (page 193)
9. Explain the impact of a single asset price jump on a volatility smile. (page 194)

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Estimating Market Risk Measures: An Introduction and Overview

Topic 1

Exam Focus

In this topic, the focus is on the estimation of market risk measures, such as value at risk (VaR). VaR identifies the probability that losses will be greater than a pre-specified threshold level. For the exam, be prepared to evaluate and calculate VaR using historical simulation and parametric models (both normal and lognormal return distributions). One drawback to VaR is that it does not estimate losses in the tail of the returns distribution. Expected shortfall (ES) does, however, estimate the loss in the tail (i.e., after the VaR threshold has been breached) by averaging loss levels at different confidence levels. Coherent risk measures incorporate personal risk aversion across the entire distribution and are more general than expected shortfall. Quantile-quantile (QQ) plots are used to visually inspect if an empirical distribution matches a theoretical distribution.

Estimating Returns

To better understand the material in this topic, it is helpful to recall the computations of arithmetic and geometric returns. Note that the convention when computing these returns (as well as VaR) is to quote return losses as positive values. For example, if a portfolio is expected to decrease in value by \$1 million, we use the terminology "expected loss is \$1 million" rather than "expected profit is –\$1 million."

Profit/loss data: Change in value of asset/portfolio, P_t, at the end of period t plus any interim payments, D_t.

$$P/L_t = P_t + D_t - P_{t-1}$$

Arithmetic return data: Assumption is that interim payments do not earn a return (i.e., no reinvestment). Hence, this approach is not appropriate for long investment horizons.

$$r_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t + D_t}{P_{t-1}} - 1$$

Geometric return data: Assumption is that interim payments are continuously reinvested. Note that this approach ensures that asset price can never be negative.

$$R_t = \ln\left(\frac{P_t + D_t}{P_{t-1}}\right)$$

Historical Simulation Approach

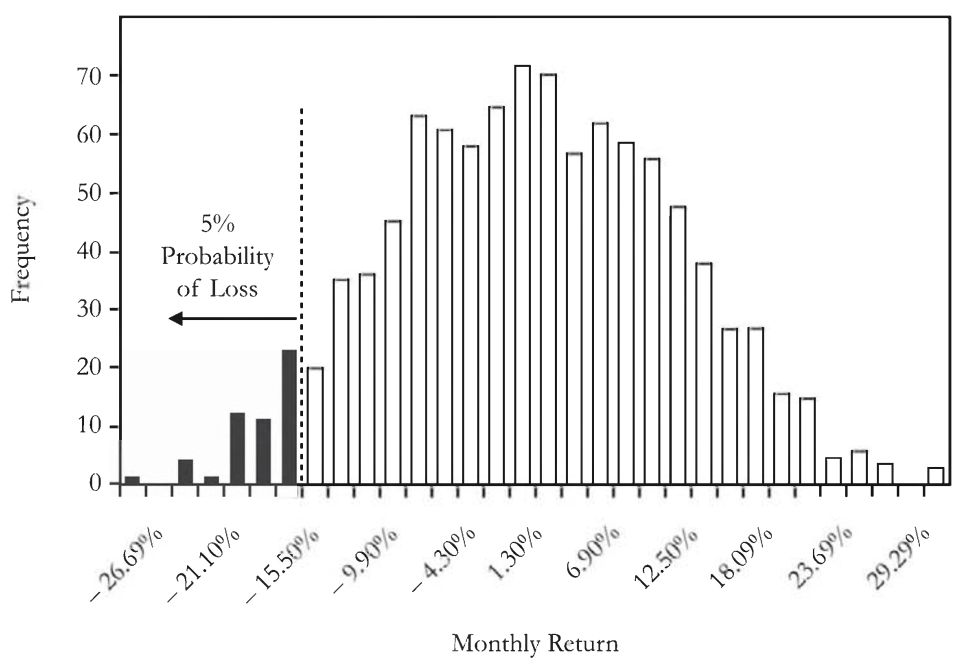
LO 1.1: Estimate VaR using a historical simulation approach.

Estimating VaR with a historical simulation approach is by far the simplest and most straightforward VaR method. To make this calculation, you simply order return observations from largest to smallest. The observation that follows the threshold loss level denotes the VaR limit. We are essentially searching for the observation that separates the tail from the body of the distribution. More generally, the observation that determines VaR for *n* observations at the $(1 - \alpha)$ confidence level would be: $(\alpha \times n) + 1$.

Professor's Note: Recall that the confidence level, $(1 - \alpha)$, is typically a large value (e.g., 95%) whereas the significance level, usually denoted as α, is much smaller (e.g., 5%).

To illustrate this VaR method, assume you have gathered 1,000 monthly returns for a security and produced the distribution shown in Figure 1. You decide that you want to compute the monthly VaR for this security at a confidence level of 95%. At a 95% confidence level, the lower tail displays the lowest 5% of the underlying distribution's returns. For this distribution, the value associated with a 95% confidence level is a return of –15.5%. If you have \$1,000,000 invested in this security, the one-month VaR is \$155,000 (–15.5% × \$1,000,000).

Figure 1: Histogram of Monthly Returns



Example: Identifying the VaR limit

Identify the ordered observation in a sample of 1,000 data points that corresponds to VaR at a 95% confidence level.

Answer:

Since VaR is to be estimated at 95% confidence, this means that 5% (i.e., 50) of the ordered observations would fall in the tail of the distribution. Therefore, the 51st ordered loss observation would separate the 5% of largest losses from the remaining 95% of returns.

Professor's Note: VaR is the quantile that separates the tail from the body of the distribution. With 1,000 observations at a 95% confidence level, there is a certain level of arbitrariness in how the ordered observations relate to VaR. In other words, should VaR be the 50th observation (i.e., $\alpha \times n$), the 51st observation [i.e., $(\alpha \times n) + 1$], or some combination of these observations? In this example, using the 51st observation was the approximation for VaR, and the method used in the assigned reading. However, on past FRM exams, VaR using the historical simulation method has been calculated as just: $(\alpha \times n)$, in this case, as the 50th observation.

Example: Computing VaR

A long history of profit/loss data closely approximates a standard normal distribution (mean equals zero; standard deviation equals one). **Estimate** the 5% VaR using the historical simulation approach.

Answer:

The VaR limit will be at the observation that separates the tail loss with area equal to 5% from the remainder of the distribution. Since the distribution is closely approximated by the standard normal distribution, the VaR is 1.65 (5% critical value from the *z*-table). Recall that since VaR is a one-tailed test, the entire significance level of 5% is in the left tail of the returns distribution.

From a practical perspective, the historical simulation approach is sensible only if you expect future performance to follow the same return generating process as in the past. Furthermore, this approach is unable to adjust for changing economic conditions or abrupt shifts in parameter values.

Parametric Estimation Approaches

LO 1.2: Estimate VaR using a parametric approach for both normal and lognormal return distributions.

In contrast to the historical simulation method, the parametric approach (e.g., the delta-normal approach) explicitly assumes a distribution for the underlying observations. For this LO, we will analyze two cases: (1) VaR for returns that follow a normal distribution, and (2) VaR for returns that follow a lognormal distribution.

Normal VaR

Intuitively, the VaR for a given confidence level denotes the point that separates the tail losses from the remaining distribution. The VaR cutoff will be in the left tail of the returns distribution. Hence, the calculated value at risk is negative, but is typically reported as a positive value since the negative amount is implied (i.e., it is the value that is at risk). In equation form, the VaR at significance level α is:

$$\text{VaR}(\alpha\%) = -\mu_{P/L} + \sigma_{P/L} \times z_{\alpha}$$

where μ and σ denote the mean and standard deviation of the profit/loss distribution and z denotes the critical value (i.e., quantile) of the standard normal. In practice, the population parameters μ and σ are not likely known, in which case the researcher will use the sample mean and standard deviation.

Example: Computing VaR (normal distribution)

Assume that the profit/loss distribution for XYZ is normally distributed with an annual mean of \$15 million and a standard deviation of \$10 million. **Calculate** the VaR at the 95% and 99% confidence levels using a parametric approach.

Answer:

VaR(5%) = –\$15 million + \$10 million × 1.65 = \$1.5 million. Therefore, XYZ expects to lose at most \$1.5 million over the next year with 95% confidence. Equivalently, XYZ expects to lose more than \$1.5 million with a 5% probability.

VaR(1%) = –\$15 million + \$10 million × 2.33 = \$8.3 million. Note that the VaR (at 99% confidence) is greater than the VaR (at 95% confidence) as follows from the definition of value at risk.

Now suppose that the data you are using is arithmetic return data rather than profit/loss data. The arithmetic returns follow a normal distribution as well. As you would expect, because of the relationship between prices, profits/losses, and returns, the corresponding VaR is very similar in format:

$$\text{VaR}(\alpha\%) = (-\mu_r + \sigma_r \times z_{\alpha}) \times P_{t-1}$$

Example: Computing VaR (arithmetic returns)

A portfolio has a beginning period value of $100. The arithmetic returns follow a normal distribution with a mean of 10% and a standard deviation of 20%. **Calculate** VaR at both the 95% and 99% confidence levels.

Answer:

$$\text{VaR}(5\%) = (-10\% + 1.65 \times 20\%) \times 100 = \$23.0$$

$$\text{VaR}(1\%) = (-10\% + 2.33 \times 20\%) \times 100 = \$36.6$$

Lognormal VaR

The lognormal distribution is right-skewed with positive outliers and bounded below by zero. As a result, the lognormal distribution is commonly used to counter the possibility of negative asset prices (P_t). Technically, if we assume that geometric returns follow a normal distribution (μ_R, σ_R), then the natural logarithm of asset prices follows a normal distribution and P_t follows a lognormal distribution. After some algebraic manipulation, we can derive the following expression for **lognormal VaR**:

$$\text{VaR}(\alpha\%) = P_{t-1} \times \left(1 - e^{\mu_R - \sigma_R \times z_\alpha}\right)$$

Example: Computing VaR (lognormal distribution)

A diversified portfolio exhibits a normally distributed geometric return with mean and standard deviation of 10% and 20%, respectively. **Calculate** the 5% and 1% lognormal VaR assuming the beginning period portfolio value is $100.

Answer:

$$\begin{aligned}\text{Lognormal VaR}(5\%) &= 100 \times (1 - \exp[0.1 - 0.2 \times 1.65]) \\ &= 100 \times (1 - \exp[-0.23]) \\ &= \$20.55\end{aligned}$$

$$\begin{aligned}\text{Lognormal VaR}(1\%) &= 100 \times (1 - \exp[0.1 - 0.2 \times 2.33]) \\ &= 100 \times (1 - \exp[-0.366]) \\ &= \$30.65\end{aligned}$$

Note that the calculation of lognormal VaR (geometric returns) and normal VaR (arithmetic returns) will be similar when we are dealing with short-time periods and practical return estimates.

Expected Shortfall

LO 1.3: Estimate the expected shortfall given P/L or return data.

A major limitation of the VaR measure is that it does not tell the investor the amount or magnitude of the actual loss. VaR only provides the maximum value we can lose for a given confidence level. The **expected shortfall** (ES) provides an estimate of the tail loss by averaging the VaRs for increasing confidence levels in the tail. Specifically, the tail mass is divided into *n* equal slices and the corresponding n – 1 VaRs are computed. For example, if n = 5, we can construct the following table based on the normal distribution:

Figure 2: Estimating Expected Shortfall

Confidence level	*VaR*	*Difference*
96%	1.7507	
97%	1.8808	0.1301
98%	2.0537	0.1729
99%	2.3263	0.2726
Average	2.003	
Theoretical true value	2.063	

Observe that the VaR increases (from *Difference* column) in order to maintain the same interval mass (of 1%) because the tails become thinner and thinner. The average of the four computed VaRs is 2.003 and represents the probability-weighted expected tail loss (a.k.a. expected shortfall). Note that as *n* increases, the expected shortfall will increase and approach the theoretical true loss [2.063 in this case; the average of a high number of VaRs (e.g., greater than 10,000)].

Estimating Coherent Risk Measures

LO 1.4: Define coherent risk measures.

LO 1.5: Estimate risk measures by estimating quantiles.

A more general risk measure than either VaR or ES is known as a coherent risk measure. A **coherent risk measure** is a weighted average of the quantiles of the loss distribution where the weights are user-specific based on individual risk aversion. ES (as well as VaR) is a special case of a coherent risk measure. When modeling the ES case, the weighting function is set to [1 / (1 – confidence level)] for all tail losses. All other quantiles will have a weight of zero.

Under expected shortfall estimation, the tail region is divided into equal probability slices and then multiplied by the corresponding quantiles. Under the more general coherent risk measure, the entire distribution is divided into equal probability slices weighted by the more general risk aversion (weighting) function.

This procedure is illustrated for n = 10. First, the entire return distribution is divided into nine (i.e., n – 1) equal probability mass slices at 10%, 20%, …, 90% (i.e., loss quantiles). Each breakpoint corresponds to a different quantile. For example, the 10% quantile (confidence level = 10%) relates to –1.2816, the 20% quantile (confidence level = 20%) relates to –0.8416, and the 90% quantile (confidence level = 90%) relates to 1.2816. Next, each quantile is weighted by the specific risk aversion function and then averaged to arrive at the value of the coherent risk measure.

This coherent risk measure is more sensitive to the choice of *n* than expected shortfall, but will converge to the risk measure's true value for a sufficiently large number of observations. The intuition is that as *n* increases, the quantiles will be further into the tails where more extreme values of the distribution are located.

LO 1.6: Evaluate estimators of risk measures by estimating their standard errors.

Sound risk management practice reminds us that estimators are only as useful as their precision. That is, estimators that are less precise (i.e., have large standard errors and wide confidence intervals) will have limited practical value. Therefore, it is best practice to also compute the standard error for all coherent risk measures.

Professor's Note: The process of estimating standard errors for estimators of coherent risk measures is quite complex, so your focus should be on interpretation of this concept.

First, let's start with a sample size of *n* and arbitrary bin width of *h* around quantile, *q*. Bin width is just the width of the intervals, sometimes called "bins," in a histogram. Computing standard error is done by realizing that the square root of the variance of the quantile is equal to the standard error of the quantile. After finding the standard error, a confidence interval for a risk measure such as VaR can be constructed as follows:

$$[q + se(q) \times z_{\alpha}] > \text{VaR} > [q - se(q) \times z_{\alpha}]$$

Example: Estimating standard errors

Construct a 90% confidence interval for 5% VaR (the 95% quantile) drawn from a standard normal distribution. Assume bin width = 0.1 and that the sample size is equal to 500.

Answer:

The quantile value, q, corresponds to the 5% VaR which occurs at 1.65 for the standard normal distribution. The confidence interval takes the following form:

$$[1.65 + 1.65 \times se(q)] > VaR > [1.65 - 1.65 \times se(q)]$$

Professor's Note: Recall that a confidence interval is a two-tailed test (unlike VaR), so a 90% confidence level will have 5% in each tail. Given that this is equivalent to the 5% significance level of VaR, the critical values of 1.65 will be the same in both cases.

Since bin width is 0.1, q is in the range 1.65 ± 0.1/2 = [1.7, 1.6]. Note that the left tail probability, p, is the area to the left of –1.7 for a standard normal distribution.

Next, calculate the probability mass between [1.7, 1.6], represented as $f(q)$. From the standard normal table, the probability of a loss *greater* than 1.7 is 0.045 (left tail). Similarly, the probability of a loss *less* than 1.6 (right tail) is 0.945. Collectively, f(q) = 1 – 0.045 – 0.945 = 0.01

The standard error of the quantile is derived from the variance approximation of q and is equal to:

$$se(q) = \frac{\sqrt{p(1-p)/n}}{f(q)}$$

Now we are ready to substitute in the variance approximation to calculate the confidence interval for VaR:

$$\left[1.65 + 1.65\frac{\sqrt{0.045(1-0.045)/500}}{0.01}\right] > VaR > \left[1.65 - 1.65\frac{\sqrt{0.045(1-0.045)/500}}{0.01}\right]$$
$$= 3.18 > VaR > 0.12$$

Let's return to the variance approximation and perform some basic comparative statistics. What happens if we increase the sample size holding all other factors constant? Intuitively, the larger the sample size the smaller the standard error and the narrower the confidence interval.

Now suppose we increase the bin size, h, holding all else constant. This will increase the probability mass $f(q)$ and reduce p, the probability in the left tail. The standard error will decrease and the confidence interval will again narrow.

Lastly, suppose that p increases indicating that tail probabilities are more likely. Intuitively, the estimator becomes less precise and standard errors increase, which widens the confidence interval. Note that the expression $p(1 - p)$ will be maximized at $p = 0.5$.

The above analysis was based on one quantile of the loss distribution. Just as the previous section generalized the expected shortfall to the coherent risk measure, we can do the same for the standard error computation. Thankfully, this complex process is not the focus of the LO.

Quantile-Quantile Plots

LO 1.7: Interpret QQ plots to identify the characteristics of a distribution.

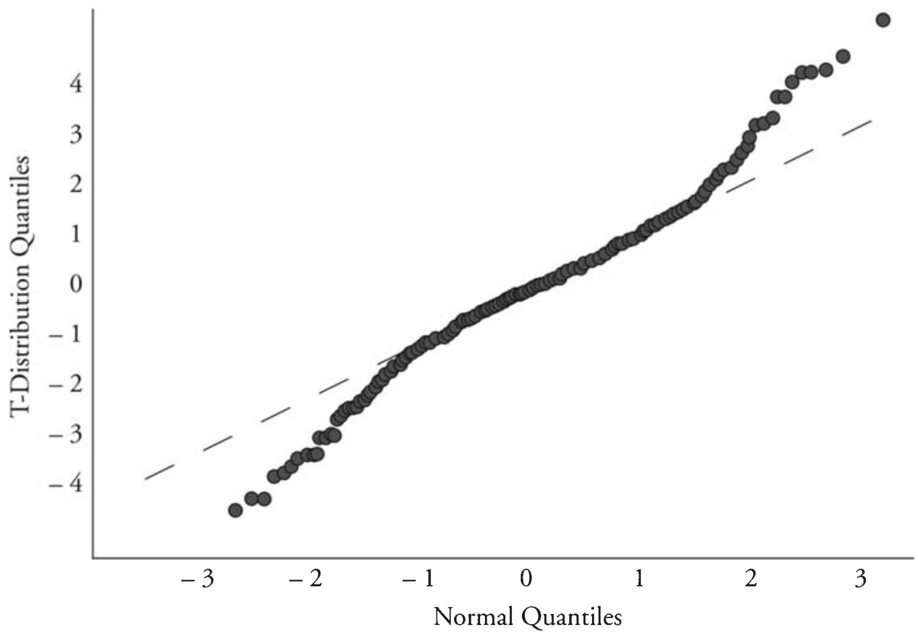
A natural question to ask in the course of our analysis is, "From what distribution is the data drawn?" The truth is that you will never really know since you only observe the realizations from random draws of an unknown distribution. However, visual inspection can be a very simple but powerful technique.

In particular, the **quantile-quantile** (QQ) **plot** is a straightforward way to visually examine if empirical data fits the reference or hypothesized theoretical distribution (assume standard normal distribution for this discussion). The process graphs the quantiles at regular confidence intervals for the empirical distribution against the theoretical distribution. As an example, if both the empirical and theoretical data are drawn from the same distribution, then the median (confidence level = 50%) of the empirical distribution would plot very close to zero, while the median of the theoretical distribution would plot exactly at zero.

Continuing in this fashion for other quantiles (40%, 60%, and so on) will map out a function. If the two distributions are very similar, the resulting QQ plot will be linear.

Let us compare a theoretical standard normal distribution relative to an empirical t-distribution (assume that the degrees of freedom for the t-distribution are sufficiently small and that there are noticeable differences from the normal distribution). We know that both distributions are symmetric, but the t-distribution will have fatter tails. Hence, the quantiles near zero (confidence level = 50%) will match up quite closely. As we move further into the tails, the quantiles between the t-distribution and the normal will diverge (see Figure 3). For example, at a confidence level of 95%, the critical z-value is –1.65, but for the t-distribution, it is closer to –1.68 (degrees of freedom of approximately 40). At 97.5% confidence, the difference is even larger, as the z-value is equal to –1.96 and the t-stat is equal to –2.02. More generally, if the middles of the QQ plot match up, but the tails do not, then the empirical distribution can be interpreted as symmetric with tails that differ from a normal distribution (either fatter or thinner).

Figure 3: QQ Plot



Key Concepts

LO 1.1

Historical simulation is the easiest method to estimate value at risk. All that is required is to reorder the profit/loss observations in increasing magnitude of losses and identify the breakpoint between the tail region and the remainder of distribution.

LO 1.2

Parametric estimation of VaR requires a specific distribution of prices or equivalently, returns. This method can be used to calculate VaR with either a normal distribution or a lognormal distribution.

Under the assumption of a normal distribution, VaR (i.e., delta-normal VaR) is calculated as follows:

$$\text{VaR} = -\mu_{P/L} + \sigma_{P/L} \times z_{\alpha}$$

Under the assumption of a lognormal distribution, lognormal VaR is calculated as follows:

$$\text{VaR} = P_{t-1} \times \left(1 - e^{\mu_R - \sigma_R \times z_{\alpha}}\right)$$

LO 1.3

VaR identifies the lower bound of the profit/loss distribution, but it does not estimate the expected tail loss. Expected shortfall overcomes this deficiency by dividing the tail region into equal probability mass slices and averaging their corresponding VaRs.

LO 1.4

A more general risk measure than either VaR or ES is known as a coherent risk measure.

LO 1.5

A coherent risk measure is a weighted average of the quantiles of the loss distribution where the weights are user-specific based on individual risk aversion. A coherent risk measure will assign each quantile (not just tail quantiles) a weight. The average of the weighted VaRs is the estimated loss.

LO 1.6

Sound risk management requires the computation of the standard error of a coherent risk measure to estimate the precision of the risk measure itself. The simplest method creates a confidence interval around the quantile in question. To compute standard error, it is necessary to find the variance of the quantile, which will require estimates from the underlying distribution.

LO 1.7

The quantile-quantile (QQ) plot is a visual inspection of an empirical quantile relative to a hypothesized theoretical distribution. If the empirical distribution closely matches the theoretical distribution, the QQ plot would be linear.

Concept Checkers

1. The VaR at a 95% confidence level is estimated to be 1.56 from a historical simulation of 1,000 observations. Which of the following statements is most likely true?
   A. The parametric assumption of normal returns is correct.
   B. The parametric assumption of lognormal returns is correct.
   C. The historical distribution has fatter tails than a normal distribution.
   D. The historical distribution has thinner tails than a normal distribution.

2. Assume the profit/loss distribution for XYZ is normally distributed with an annual mean of $20 million and a standard deviation of $10 million. The 5% VaR is calculated and interpreted as which of the following statements?
   A. 5% probability of losses of at least $3.50 million.
   B. 5% probability of earnings of at least $3.50 million.
   C. 95% probability of losses of at least $3.50 million.
   D. 95% probability of earnings of at least $3.50 million.

3. Which of the following statements about expected shortfall estimates and coherent risk measures are true?
   A. Expected shortfall and coherent risk measures estimate quantiles for the entire loss distribution.
   B. Expected shortfall and coherent risk measures estimate quantiles for the tail region.
   C. Expected shortfall estimates quantiles for the tail region and coherent risk measures estimate quantiles for the non-tail region only.
   D. Expected shortfall estimates quantiles for the entire distribution and coherent risk measures estimate quantiles for the tail region only.

4. Which of the following statements most likely increases standard errors from coherent risk measures?
   A. Increasing sample size and increasing the left tail probability.
   B. Increasing sample size and decreasing the left tail probability.
   C. Decreasing sample size and increasing the left tail probability.
   D. Decreasing sample size and decreasing the left tail probability.

5. The quantile-quantile plot is best used for what purpose?
   A. Testing an empirical distribution from a theoretical distribution.
   B. Testing a theoretical distribution from an empirical distribution.
   C. Identifying an empirical distribution from a theoretical distribution.
   D. Identifying a theoretical distribution from an empirical distribution.

Concept Checker Answers

1. **D** The historical simulation indicates that the 5% tail loss begins at 1.56, which is less than the 1.65 predicted by a standard normal distribution. Therefore, the historical simulation has thinner tails than a standard normal distribution.

2. **D** The value at risk calculation at 95% confidence is: –20 million + 1.65 × 10 million = –\$3.50 million. Since the expected loss is negative and VaR is an implied negative amount, the interpretation is that XYZ will earn less than +\$3.50 million with 5% probability, which is equivalent to XYZ earning at least \$3.50 million with 95% probability.

3. **B** ES estimates quantiles for n – 1 equal probability masses in the tail region only. The coherent risk measure estimates quantiles for the entire distribution including the tail region.

4. **C** Decreasing sample size clearly increases the standard error of the coherent risk measure given that standard error is defined as:

$$se(q) = \frac{\sqrt{p(1-p)/n}}{f(q)}$$

As the left tail probability, p, increases, the probability of tail events increases, which also increases the standard error. Mathematically, $p(1 - p)$ increases as p increases until $p = 0.5$. Small values of p imply smaller standard errors.

5. **C** Once a sample is obtained, it can be compared to a reference distribution for possible identification. The QQ plot maps the quantiles one to one. If the relationship is close to linear, then a match for the empirical distribution is found. The QQ plot is used for visual inspection only without any formal statistical test.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Non-parametric Approaches

Topic 2

Exam Focus

This topic introduces non-parametric estimation and bootstrapping (i.e., resampling). The key difference between these approaches and parametric approaches discussed in the previous topic is that with non-parametric approaches the underlying distribution is not specified, and it is a data driven, not assumption driven, analysis. For example, historical simulation is limited by the discreteness of the data, but non-parametric analysis "smoothes" the data points to allow for any VaR confidence level between observations. For the exam, pay close attention to the description of the bootstrap historical simulation approach as well as the various weighted historical simulations approaches.

Non-parametric estimation does not make restrictive assumptions about the underlying distribution like parametric methods, which assume very specific forms such as normal or lognormal distributions. Non-parametric estimation lets the data drive the estimation. The flexibility of these methods makes them excellent candidates for VaR estimation, especially if tail events are sparse.

Bootstrap Historical Simulation Approach

LO 2.1: Apply the bootstrap historical simulation approach to estimate coherent risk measures.

The **bootstrap historical simulation** is a simple and intuitive estimation procedure. In essence, the bootstrap technique draws a sample from the original data set, records the VaR from that particular sample and "returns" the data. This procedure is repeated over and over and records multiple sample VaRs. Since the data is always "returned" to the data set, this procedure is akin to sampling with replacement. The best VaR estimate from the full data set is the average of all sample VaRs.

This same procedure can be performed to estimate the expected shortfall (ES). Each drawn sample will calculate its own ES by slicing the tail region into *n* slices and averaging the VaRs at each of the n – 1 quantiles. This is exactly the same procedure described in the previous topic. Similarly, the best estimate of the expected shortfall for the original data set is the average of all of the sample expected shortfalls.

Empirical analysis demonstrates that the bootstrapping technique consistently provides more precise estimates of coherent risk measures than historical simulation on raw data alone.

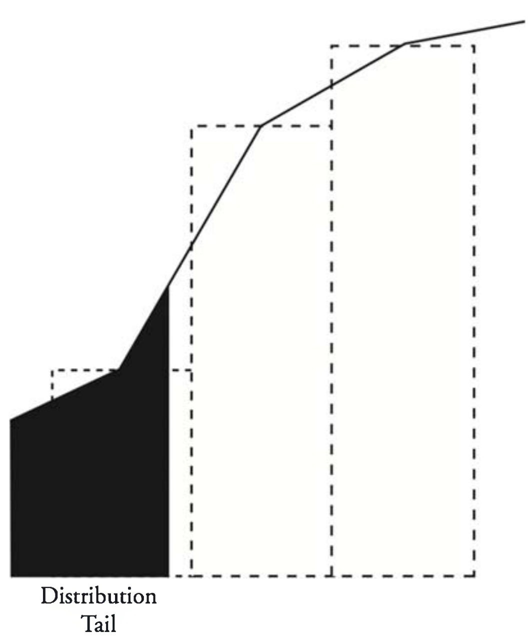
Using Non-Parametric Estimation

LO 2.2: Describe historical simulation using non-parametric density estimation.

The clear advantage of the traditional historical simulation approach is its simplicity. One obvious drawback, however, is that the discreteness of the data does not allow for estimation of VaRs between data points. If there were 100 historical observations, then it is straightforward to estimate VaR at the 95% or the 96% confidence levels, and so on. However, this method is unable to incorporate a confidence level of 95.5%, for example. More generally, with *n* observations, the historical simulation method only allows for *n* different confidence levels.

One of the advantages of non-parametric density estimation is that the underlying distribution is free from restrictive assumptions. Therefore, the existing data points can be used to "smooth" the data points to allow for VaR calculation at all confidence levels. The simplest adjustment is to connect the midpoints between successive histogram bars in the original data set's distribution. See Figure 1 for an illustration of this **surrogate density function.** Notice that by connecting the midpoints, the lower bar "receives" area from the upper bar, which "loses" an equal amount of area. In total, no area is lost, only displaced, so we still have a probability distribution function, just with a modified shape. The shaded area in Figure 1 represents a possible confidence interval, which can be utilized regardless of the size of the data set. The major improvement of this non-parametric approach over the traditional historical simulation approach is that VaR can now be calculated for a continuum of points in the data set.

Figure 1: Surrogate Density Function



Following this logic, one can see that the linear adjustment is a simple solution to the interval problem. A more complicated adjustment would involve connecting curves, rather than lines, between successive bars to better capture the characteristics of the data.

Weighted Historical Simulation Approaches

LO 2.3: Compare and contrast the age-weighted, the volatility-weighted, the correlation-weighted, and the filtered historical simulation approaches.

The previous weighted historical simulation, discussed in Topic 1, assumed that both current and past (arbitrary) *n* observations up to a specified cutoff point are used when computing the current period VaR. Older observations beyond the cutoff date are assumed to have a zero weight and the relevant *n* observations have equal weight of (1 / n). While simple in construction, there are obvious problems with this method. Namely, why is the *n*th observation as important as all other observations, but the *(n + 1)*th observation is so unimportant that it carries no weight? Current VaR may have "ghost effects" of previous events that remain in the computation until they disappear (after *n* periods). Furthermore, this method assumes that each observation is independent and identically distributed. This is a very strong assumption, which is likely violated by data with clear seasonality (i.e., seasonal volatility). This topic identifies four improvements to the traditional historical simulation method.

Age-weighted Historical Simulation

The obvious adjustment to the equal-weighted assumption used in historical simulation is to weight recent observations more and distant observations less. One method proposed by Boudoukh, Richardson, and Whitelaw is as follows.[1] Assume w(1) is the probability weight for the observation that is one day old. Then w(2) can be defined as λw(1), w(3) can be defined as λ^2w(1), and so on. The decay parameter, λ, can take on values $0 \leq \lambda \leq 1$ where values close to 1 indicate slow decay. Since all of the weights must sum to 1, we conclude that $w(1) = (1 - \lambda) / (1 - \lambda^n)$. More generally, the weight for an observation that is *i* days old is equal to:

$$w(i) = \frac{\lambda^{i-1}(1-\lambda)}{1-\lambda^n}$$

The implication of the age-weighted simulation is to reduce the impact of ghost effects and older events that may not reoccur. Note that this more general weighting scheme suggests that historical simulation is a special case where $\lambda = 1$ (i.e., no decay) over the estimation window.

Professor's Note: This approach is also known as the hybrid approach.

1. Boudoukh, J., M. Richardson, and R. Whitelaw. 1998. "The best of both worlds: a hybrid approach to calculating value at risk." *Risk* 11: 64–67.

Volatility-weighted Historical Simulation

Another approach is to weight the individual observations by volatility rather than proximity to the current date. This was introduced by Hull and White to incorporate changing volatility in risk estimation.[2] The intuition is that if recent volatility has increased, then using historical data will underestimate the current risk level. Similarly, if current volatility is markedly reduced, the impact of older data with higher periods of volatility will overstate the current risk level.

This process is captured in the expression below for estimating VaR on day T. The expression is achieved by adjusting each daily return, $r_{t,i}$ on day t upward or downward based on the then-current volatility forecast, $\sigma_{t,i}$ (estimated from a GARCH or EWMA model) relative to the current volatility forecast on day T.

$$r^*_{t,i} = \left(\frac{\sigma_{T,i}}{\sigma_{t,i}}\right) r_{t,i}$$

where:
$r_{t,i}$ = actual return for asset i on day t
$\sigma_{t,i}$ = volatility forecast for asset i on day t (made at the end of day t − 1)
$\sigma_{T,i}$ = current forecast of volatility for asset i

Thus, the volatility-adjusted return, $r^*_{t,i}$, is replaced with a larger (smaller) expression if current volatility exceeds (is below) historical volatility on day i. Now, VaR, ES, and any other coherent risk measure can be calculated in the usual way after substituting historical returns with volatility-adjusted returns.

There are several advantages of the volatility-weighted method. First, it explicitly incorporates volatility into the estimation procedure in contrast to other historical methods. Second, the near-term VaR estimates are likely to be more sensible in light of current market conditions. Third, the volatility-adjusted returns allow for VaR estimates that are higher than estimates with the historical data set.

Correlation-weighted Historical Simulation

As the name suggests, this methodology incorporates updated correlations between asset pairs. This procedure is more complicated than the volatility-weighting approach, but it follows the same basic principles. Since the corresponding LO does not require calculations, the exact matrix algebra would only complicate our discussion. Intuitively, the historical correlation (or equivalently variance-covariance) matrix needs to be adjusted to the new information environment. This is accomplished, loosely speaking, by "multiplying" the historic returns by the revised correlation matrix to yield updated correlation-adjusted returns.

2. Hull, J., and A. White. 1998. "Incorporating volatility updating into the historical simulation method for value-at-risk." *Journal of Risk* 1: 5–19.

Let us look at the variance-covariance matrix more closely. In particular, we are concerned with diagonal elements and the off-diagonal elements. The off-diagonal elements represent the current covariance between asset pairs. On the other hand, the diagonal elements represent the updated variances (covariance of the asset return with itself) of the individual assets.

$$\Sigma = \begin{pmatrix} \sigma_{i,i} & \sigma_{i,j} \\ \sigma_{j,i} & \sigma_{j,j} \end{pmatrix} = \begin{pmatrix} \text{Variance}(X_i) & \text{Cov}\left(X_i, X_j\right) \\ \text{Cov}\left(X_j, X_i\right) & \text{Variance}(X_j) \end{pmatrix}$$

Notice that updated variances were utilized in the previous approach as well. Thus, correlation-weighted simulation is an even richer analytical tool than volatility-weighted simulation because it allows for updated variances (volatilities) as well as covariances (correlations).

Filtered Historical Simulation

The filtered historical simulation is the most comprehensive, and hence most complicated, of the non-parametric estimators. The process combines the historical simulation model with conditional volatility models (like GARCH or asymmetric GARCH). Thus, the method contains both the attractions of the traditional historical simulation approach with the sophistication of models that incorporate changing volatility. In simplified terms, the model is flexible enough to capture conditional volatility and volatility clustering as well as a surprise factor that could have an asymmetric effect on volatility.

The model will forecast volatility for each day in the sample period and the volatility will be standardized by dividing by realized returns. Bootstrapping is used to simulate returns which incorporate the current volatility level. Finally, the VaR is identified from the simulated distribution. The methodology can be extended over longer holding periods or for multi-asset portfolios.

In sum, the filtered historical simulation method uses bootstrapping and combines the traditional historical simulation approach with rich volatility modeling. The results are then sensitive to changing market conditions and can predict losses outside the historical range. From a computational standpoint, this method is very reasonable even for large portfolios, and empirical evidence supports its predictive ability.

Advantages and Disadvantages of Non-Parametric Methods

LO 2.4: Identify advantages and disadvantages of non-parametric estimation methods.

Any risk manager should be prepared to use non-parametric estimation techniques. There are some clear advantages to non-parametric methods, but there is some danger as well. Therefore, it is incumbent to understand the advantages, the disadvantages, and the appropriateness of the methodology for analysis.

Advantages of non-parametric methods include the following:

- Intuitive and often computationally simple (even on a spreadsheet).
- Not hindered by parametric violations of skewness, fat-tails, et cetera.
- Avoids complex variance-covariance matrices and dimension problems.
- Data is often readily available and does not require adjustments (e.g., financial statements adjustments).
- Can accommodate more complex analysis (e.g., by incorporating age-weighting with volatility-weighting).

Disadvantages of non-parametric methods include the following:

- Analysis depends critically on historical data.
- Volatile data periods lead to VaR and ES estimates that are too high.
- Quiet data periods lead to VaR and ES estimates that are too low.
- Difficult to detect structural shifts/regime changes in the data.
- Cannot accommodate plausible large impact events if they did not occur within the sample period.
- Difficult to estimate losses significantly larger than the maximum loss within the data set (historical simulation cannot; volatility-weighting can, to some degree).
- Need sufficient data, which may not be possible for new instruments or markets.

Key Concepts

LO 2.1

Bootstrapping involves resampling a subset of the original data set with replacement. Each draw (subsample) yields a coherent risk measure (VaR or ES). The average of the risk measures across all samples is then the best estimate.

LO 2.2

The discreteness of historical data reduces the number of possible VaR estimates since historical simulation cannot adjust for significance levels between ordered observations. However, non-parametric density estimation allows the original histogram to be modified to fill in these gaps. The process connects the midpoints between successive columns in the histogram. The area is then "removed" from the upper bar and "placed" in the lower bar, which creates a "smooth" function between the original data points.

LO 2.3

One important limitation to the historical simulation method is the equal-weight assumed for all data in the estimation period, and zero weight otherwise. This arbitrary methodology can be improved by using age-weighted simulation, volatility-weighted simulation, correlation-weighted simulation, and filtered historical simulation.

The age-weighted simulation method adjusts the most recent (distant) observations to be more (less) heavily weighted.

The volatility-weighting procedure incorporates the possibility that volatility may change over the estimation period, which may understate or overstate current risk by including stale data. The procedure replaces historic returns with volatility-adjusted returns; however, the actual procedure of estimating VaR is unchanged (i.e., only the data inputs change).

Correlation-weighted simulation updates the variance-covariance matrix between the assets in the portfolio. The off-diagonal elements represent the covariance pairs while the diagonal elements update the individual variance estimates. Therefore, the correlation-weighted methodology is more general than the volatility-weighting procedure by incorporating both variance and covariance adjustments.

Filtered historical simulation is the most complex estimation method. The procedure relies on bootstrapping of standardized returns based on volatility forecasts. The volatility forecasts arise from GARCH or similar models and are able to capture conditional volatility, volatility clustering, and/or asymmetry.

LO 2.4

Advantages of non-parametric models include: data can be skewed or have fat tails; they are conceptually straightforward; there is readily available data; and they can accommodate more complex analysis. Disadvantages focus mainly on the use of historical data, which limits the VaR forecast to (approximately) the maximum loss in the data set; they are slow to respond to changing market conditions; they are affected by volatile (quiet) data periods; and they cannot accommodate plausible large losses if not in the data set.

Concept Checkers

1. Johanna Roberto has collected a data set of 1,000 daily observations on equity returns. She is concerned about the appropriateness of using parametric techniques as the data appears skewed. Ultimately, she decides to use historical simulation and bootstrapping to estimate the 5% VaR. Which of the following steps is most likely to be part of the estimation procedure?
   A. Filter the data to remove the obvious outliers.
   B. Repeated sampling with replacement.
   C. Identify the tail region from reordering the original data.
   D. Apply a weighting procedure to reduce the impact of older data.

2. All of the following approaches improve the traditional historical simulation approach for estimating VaR except the:
   A. volatility-weighted historical simulation.
   B. age-weighted historical simulation.
   C. market-weighted historical simulation.
   D. correlation-weighted historical simulation.

3. Which of the following statements about age-weighting is most accurate?
   A. The age-weighting procedure incorporates estimates from GARCH models.
   B. If the decay factor in the model is close to 1, there is persistence within the data set.
   C. When using this approach, the weight assigned on day *i* is equal to: $w(i) = \lambda^{i-1} \times (1-\lambda)/(1-\lambda^{i})$.
   D. The number of observations should at least exceed 250.

4. Which of the following statements about volatility-weighting is true?
   A. Historic returns are adjusted, and the VaR calculation is more complicated.
   B. Historic returns are adjusted, and the VaR calculation procedure is the same.
   C. Current period returns are adjusted, and the VaR calculation is more complicated.
   D. Current period returns are adjusted, and the VaR calculation is the same.

5. All of the following items are generally considered advantages of non-parametric estimation methods except:
   A. ability to accommodate skewed data.
   B. availability of data.
   C. use of historical data.
   D. little or no reliance on covariance matrices.

Concept Checker Answers

1. **B** Bootstrapping from historical simulation involves repeated sampling with replacement. The 5% VaR is recorded from each sample draw. The average of the VaRs from all the draws is the VaR estimate. The bootstrapping procedure does not involve filtering the data or weighting observations. Note that the VaR from the original data set is not used in the analysis.

2. **C** Market-weighted historical simulation is not discussed in this topic. Age-weighted historical simulation weights observations higher when they appear closer to the event date. Volatility-weighted historical simulation adjusts for changing volatility levels in the data. Correlation-weighted historical simulation incorporates anticipated changes in correlation between assets in the portfolio.

3. **B** If the intensity parameter (i.e., decay factor) is close to 1, there will be persistence (i.e., slow decay) in the estimate. The expression for the weight on day i has i in the exponent when it should be n. While a large sample size is generally preferred, some of the data may no longer be representative in a large sample.

4. **B** The volatility-weighting method adjusts historic returns for current volatility. Specifically, return at time t is multiplied by (current volatility estimate / volatility estimate at time t). However, the actual procedure for calculating VaR using a historical simulation method is unchanged; it is only the inputted data that changes.

5. **C** The use of historical data in non-parametric analysis is a disadvantage, not an advantage. If the estimation period was quiet (volatile) then the estimated risk measures may understate (overstate) the current risk level. Generally, the largest VaR cannot exceed the largest loss in the historical period. On the other hand, the remaining choices are all considered advantages of non-parametric methods. For instance, the non-parametric nature of the analysis can accommodate skewed data, data points are readily available, and there is no requirement for estimates of covariance matrices.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Backtesting VaR

Topic 3

Exam Focus

We use value at risk (VaR) methodologies to model risk. With VaR models, we seek to approximate the changes in value that our portfolio would experience in response to changes in the underlying risk factors. Model validation incorporates several methods that we use in order to determine how close our approximations are to actual changes in value. Through model validation, we are able to determine what confidence to place in our models, and we have the opportunity to improve their accuracy. For the exam, be prepared to validate approaches that measure how close VaR model approximations are to actual changes in value. Also, understand how the log-likelihood ratio (LR) is used to test the validity of VaR models for Type I and Type II errors for both unconditional and conditional tests. Finally, be familiar with Basel Committee outcomes that require banks to backtest their internal VaR models and penalize banks by enforcing higher capital requirements for excessive exceptions.

Backtesting VaR Models

LO 3.1: Define backtesting and exceptions and explain the importance of backtesting VaR models.

Backtesting is the process of comparing losses predicted by a value at risk (VaR) model to those actually experienced over the testing period. It is an important tool for providing *model validation*, which is a process for determining whether a VaR model is adequate. The main goal of backtesting is to ensure that actual losses do not exceed expected losses at a given confidence level. The number of actual observations that fall outside a given confidence level are called *exceptions*. The number of exceptions falling outside of the VaR confidence level should not exceed one minus the confidence level. For example, exceptions should occur less than 5% of the time if the confidence level is 95%.

Backtesting is extremely important for risk managers and regulators to validate whether VaR models are properly calibrated or accurate. If the level of exceptions is too high, models should be recalibrated and risk managers should re-evaluate assumptions, parameters, and/or modeling processes. The Basel Committee allows banks to use internal VaR models to measure their risk levels, and backtesting provides a critical evaluation technique to test the adequacy of those internal VaR models. Bank regulators rely on backtesting to verify risk models and identify banks that are designing models that underestimate their risk. Banks with excessive exceptions (more than four exceptions in a sample size of 250) are penalized with higher capital requirements.

LO 3.2: Explain the significant difficulties in backtesting a VaR model.

VaR models are based on static portfolios, while actual portfolio compositions are constantly changing as relative prices change and positions are bought and sold. Multiple risk factors affect actual profit and loss, but they are not included in the VaR model. For example, the actual returns are complicated by intraday changes as well as profit and loss factors that result from commissions, fees, interest income, and bid-ask spreads. Such effects can be minimized by backtesting with a relatively short time horizon such as a daily holding period.

Another difficulty with backtesting is that the sample backtested may not be representative of the true underlying risk. The backtesting period constitutes a limited sample, so we do not expect to find the predicted number of exceptions in every sample. At some level, we must reject the model, which suggests the need to find an acceptable level of exceptions.

Risk managers should track both actual and hypothetical returns that reflect VaR expectations. The VaR modeled returns are comparable to the hypothetical return that would be experienced had the portfolio remained constant for the holding period. Generally, we compare the VaR model returns to **cleaned returns** (i.e., actual returns adjusted for all changes that arise from changes that are not marked to market, like funding costs and fee income). Both actual and hypothetical returns should be backtested to verify the validity of the VaR model, and the VaR modeling methodology should be adjusted if hypothetical returns fail when backtesting.

Using Failure Rates in Model Verification

LO 3.3: Verify a model based on exceptions or failure rates.

If a VaR model were completely accurate, we would expect VaR loss limits to be exceeded (this is called an **exception**) with the same frequency predicted by the confidence level used in the VaR model. For example, if we use a 95% confidence level, we expect to find exceptions in 5% of instances. Thus, backtesting is the process of systematically comparing actual (exceptions) and predicted loss levels.

The backtesting period constitutes a limited sample at a specific confidence level. We would not expect to find the predicted number of exceptions in every sample. How, then, do we determine if the actual number of exceptions is acceptable? If we expect five exceptions and find eight, is that too many? What about nine? At some level, we must reject the model, and we need to know that level.

Failure rates define the percentage of times the VaR confidence level is exceeded in a given sample. Under Basel rules, bank VaR models must use a 99% confidence level, which means a bank must report the VaR amount at the 1% left-tail level for a total of T days. The total number of times exceptions occur is computed as N (the sum of the number of times actual returns exceeded the previous day's VaR amount).

An unbiased measure of the number of exceptions as a proportion of the number of samples is called the **failure rate**. The probability of exception, p, equals one minus the confidence

level ($p = 1 - c$). If we use N to represent the number of exceptions and T to represent the sample size, the failure rate is computed as N / T. This failure rate is unbiased if the computed p approaches the confidence level as the sample size increases. Nonparametric tests can then be used to see if the number of times a VaR model fails is acceptable or not.

Example: Computing the probability of exception

Suppose a VaR of $10 million is calculated at a 95% confidence level. What is an acceptable probability of exception for exceeding this VaR amount?

Answer:

We expect to have exceptions (i.e., losses exceeding $10 million) 5% of the time (1 – 95%). If exceptions are occurring with greater frequency, we may be underestimating the actual risk. If exceptions are occurring less frequently, we may be overestimating risk and misallocating capital as a result.

Testing that the model is correctly calibrated requires the calculation of a *z*-score, where *x* is the number of actual exceptions observed. This *z*-score is then compared to the critical value at the chosen level of confidence (e.g., 1.96 for the 95% confidence level) to determine whether the VaR model is unbiased.

$$z = \frac{x - pT}{\sqrt{p(1-p)T}}$$

Example: Model verification

Suppose daily revenue fell below a predetermined VaR level (at the 95% confidence level) on 22 days during a 252-day period. Is this sample an unbiased sample?

Answer:

To answer this question, we calculate the *z*-score as follows:

$$z = \frac{22 - 0.05(252)}{\sqrt{0.05(0.95)252}} = \frac{22 - 12.6}{\sqrt{11.97}} = \frac{9.4}{3.4598} = 2.72$$

Based on the calculation, this is not an unbiased sample because the computed *z*-value of 2.72 is larger than the 1.96 critical value at the 95% confidence level. In this case, we would reject the null hypothesis that the VaR model is unbiased and conclude that the maximum number of exceptions has been exceeded.

Note that the confidence level at which we choose to reject or fail to reject a model is not related to the confidence level at which VaR was calculated. In evaluating the accuracy of the model, we are comparing the number of exceptions observed with the maximum number of exceptions that would be expected from a correct model at a given confidence level.

Type I and Type II Errors

LO 3.4: Define and identify Type I and Type II errors.

A sample cannot be used to determine with absolute certainty whether the model is accurate. However, we can determine the accuracy of the model and the probability of having the number of exceptions that we experienced. When determining a range for the number of exceptions that we would accept, we must strike a balance between the chances of *rejecting an accurate model* (**Type I error**) and the chances of *failing to reject an inaccurate model* (**Type II error**). The model verification test involves a tradeoff between Type I and Type II errors. The goal in backtesting is to create a VaR model with a low Type I error and include a test for a very low Type II error rate. We can establish such ranges at different confidence levels using a binomial probability distribution based on the size of the sample.

The binomial test is used to determine if the number of exceptions is acceptable at various confidence levels. Banks are required to use 250 days of data to be tested at the 99% confidence level. This results in a failure rate, or $p = 0.01$, of only 2.5 exceptions in a 250-day time horizon. Bank regulators impose a penalty in the form of higher capital requirements if five or more exceptions are observed. Figure 1 illustrates that we expect five or more exceptions 10.8% of the time given a 99% confidence level. Regulators will reject a correct model or commit a Type I error in these cases at the far right tail.

Figure 2 illustrates the far left tail of the distribution, where we evaluate Type II errors. For less than five exceptions, regulators will fail to reject an incorrect model at a 97% confidence level (rather than a 99% confidence level) 12.8% of the time. Note that if we lower the confidence level to 95%, the probability of committing Type I and Type II errors increases significantly.

Figure 1: Type I Error (Exceptions When Model Is Correct)

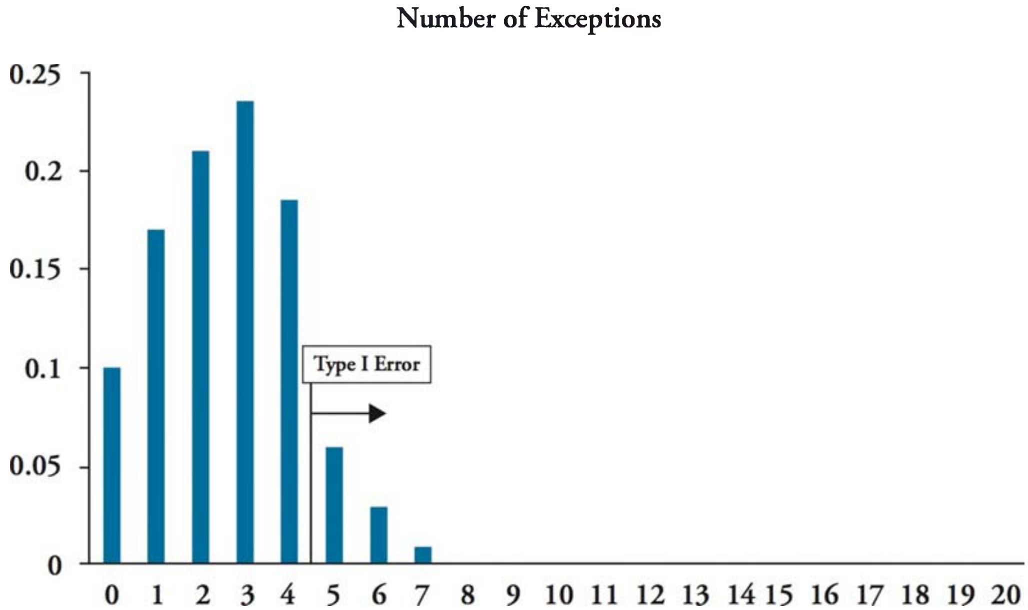
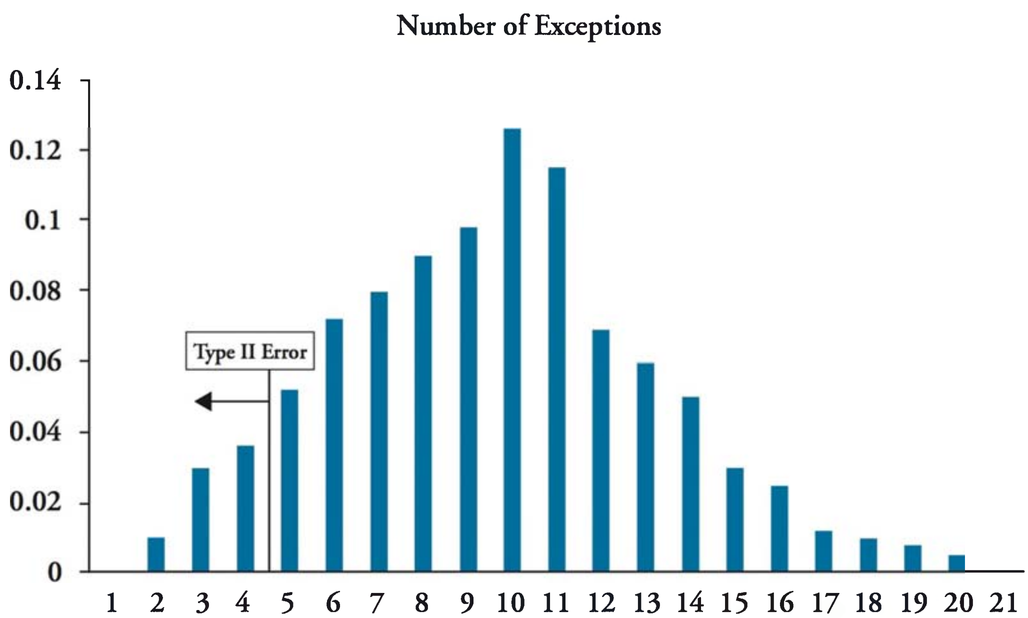


Figure 2: Type II Error (Exceptions When Model Is Incorrect)



Unconditional Coverage

Kupiec (1995)[1] determined a measure to accept or reject models using the tail points of a log-likelihood ratio (LR) as follows:

$$LR_{uc} = -2\ln[(1-p)^{T-N}p^N] + 2\ln\{[1-(N/T)]^{T-N}(N/T)^N\}$$

where p is the probability level, T is the sample size, N is the number of exceptions, and LR_{uc} is the test statistic for unconditional coverage (uc).

The term *unconditional coverage* refers to the fact that we are not concerned about independence of exception observations or the timing of when the exceptions occur. We simply are concerned with the number of total exceptions. We would *reject* the hypothesis that the model is correct if the $LR_{uc} > 3.84$. This critical LR value is used to determine the range of acceptable exceptions without rejecting the VaR model at the 95% confidence level of the log-likelihood test. Figure 3 provides the nonrejection region for the number of failures (N) based on the probability level (p), confidence level (c), and time period (T).

Figure 3: Nonrejection Regions

p	c	$T = 252$	$T = 1,000$
0.01	99.0%	$N < 7$	$4 < N < 17$
0.025	97.5%	$2 < N < 12$	$15 < N < 36$
0.05	95.0%	$6 < N < 20$	$37 < N < 65$

The LR_{uc} test could be used to backtest a daily holding period VaR model that was constructed using a 95% confidence level over a 252-day period. If the model is accurate, the expected number of exceptions will be 5% of 252, or 12.6. We know that even if the model is precise, there will be some variation in the number of exceptions between samples. The mean of the samples will approach 12.6 as the number of samples increases if the model is unbiased. However, we also know that even if the model is incorrect, we might still end up with the number of exceptions at or near 12.6.

Figure 3 can be used to illustrate how increasing the sample size allows us to reject the model more easily. For example, at the 97.5% confidence, where T = 252, the test interval is 2 / 252 = 0.0079, 12 / 252 = 0.0476. When T is increased to 1,000, the test interval shrinks to 16 / 1,000 = 0.016, 36 / 1,000 = 0.036.

Figure 3 also illustrates that it is difficult to backtest VaR models constructed with higher levels of confidence, because the number of exceptions is often not high enough to provide meaningful information. Notice that at the 95% confidence level, the test interval for T = 252 is 6 / 252 = 0.024, 20 / 252 = 0.079. With higher confidence levels (i.e., smaller values of p), the range of acceptable exceptions is smaller. Thus, it becomes difficult to determine if the model is overstating risks (i.e., fewer than expected exceptions) or if the number of exceptions is simply at the lower range of acceptable. Banks will sometimes choose to use

1. Paul Kupiec, "Techniques for Verifying the Accuracy of Risk Measurement Models," *Journal of Derivatives*, 2 (December 1995): 73–84.

a higher value of p such as 5%, in order to validate the model with a sufficient number of deviations.

Figure 4 shows the calculated values of LR_{uc} with 252-day samples for three different VaR confidence levels and various exceptions per sample. To illustrate how Figure 4 was created, the test statistic for unconditional coverage in the first row (where N = 7, T = 252, and p = 0.05) is computed as follows:

$$LR_{uc} = -2\ln[(1-0.05)^{252-7}(0.05)^{7}] + 2\ln\{[1-(7/252)]^{252-7}(7/252)^{7}\} = 3.10$$

The tail points of the unconditional log-likelihood ratio use a chi-squared distribution with one degree of freedom when T is large and the null hypothesis is that p is the true probability or true failure rate. As mentioned, the chi-squared test statistic is 3.84 at a 95% confidence level. Note that the bold areas in Figure 4 correspond to LRs greater than 3.84.

Figure 4: LR_{uc} Values for T = 252

						N						
c	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*15*	*20*
95.0%	**18.69**	**14.30**	**10.97**	**8.33**	**6.20**	**4.48**	3.10	2.02	1.20	0.61	0.45	**3.91**
97.5%	**7.03**	**4.09**	2.19	0.99	0.30	0.01	0.08	0.43	1.05	1.90	**8.94**	**19.59**
99.0%	1.20	0.12	0.09	0.75	1.92	3.50	**5.42**	**7.64**	**10.12**	**12.83**	**29.19**	**49.15**

Professor's Note: The chi-squared test statistic is the square of the normal distribution test statistic. Recall that the normal distribution test statistic at a 95% confidence level is 1.96, so squaring this value results in 3.84.

Example: Testing for unconditional coverage

Suppose that a risk manager needs to backtest a daily VaR model that was constructed using a 95% confidence level over a 252-day period. If the sample revealed 12 exceptions, should we reject or fail to reject the null hypothesis that p is the true probability of failure for this VaR model?

Answer:

We compute the test statistic as follows at the 95% confidence level (with T = 252, p = 0.05, and N = 12):

$$LR_{uc} = -2\ln[(1-0.05)^{252-12}(0.05)^{12}] + 2\ln\{[1-(12/252)]^{252-12}(12/252)^{12}\} = 0.03$$

The LR_{uc} is less than the test statistic of 3.84. Therefore, we fail to reject the null hypothesis and the model is validated based on this sample test. We would expect the number of exceptions to be 12.6 (N = 0.05 × 252 = 12.6).

Figure 4 illustrates that we would not reject the model at the 95% confidence level if the number of exceptions in our sample is greater than 6 and less than 20. For this example, if N was greater than or equal to 20, it would indicate that the VaR amount is too low and that the model understates the probability of large losses. If values of N are less than or equal to 6, it would indicate that the VaR model is too conservative.

Using VaR to Measure Potential Losses

Oftentimes, the purpose of using VaR is to measure some level of potential losses. There are two theories about choosing a holding period for this application. The first theory is that the holding period should correspond to the amount of time required to either liquidate or hedge the portfolio. Thus, VaR would calculate possible losses before corrective action could take effect. The second theory is that the holding period should be chosen to match the period over which the portfolio is not expected to change due to non-risk-related activity (e.g., trading). The two theories are not that different. For example, many banks use a daily VaR to correspond with the daily profit and loss measures. In this application, the holding period is more significant than the confidence level.

Conditional Coverage

LO 3.5: Explain the need to consider conditional coverage in the backtesting framework.

So far in the examples and discussion, we have been backtesting models based on **unconditional coverage**, in which the timing of our exceptions was not considered. Conditioning considers the time variation of the data. In addition to having a predictable number of exceptions, we also anticipate the exceptions to be fairly equally distributed across time. A bunching of exceptions may indicate that market correlations have changed or that our trading positions have been altered. In the event that exceptions are not independent, the risk manager should incorporate models that consider time variation in risk.

We need some guide to determine if the bunching is random or caused by one of these changes. By including a measure of the independence of exceptions, we can measure **conditional coverage** of the model. Christofferson[2] proposed extending the unconditional coverage test statistic (LR_{uc}) to allow for potential time variation of the data. He developed a statistic to determine the serial independence of deviations using a log-likelihood ratio test (LR_{ind}). The overall log-likelihood test statistic for conditional coverage (LR_{cc}) is then computed as:

$$\mathrm{LR_{cc}} = \mathrm{LR_{uc}} + \mathrm{LR_{ind}}$$

Each individual component is independently distributed as chi-squared, and the sum is also distributed as chi-squared. At the 95% confidence level, we would reject the model if $\mathrm{LR_{cc}} > 5.99$ and we would reject the independence term alone if $\mathrm{LR_{ind}} > 3.84$. If exceptions

2. P.F. Christofferson, "Evaluating Interval Forecasts," *International Economic Review,* 39 (1998), 841–862.

are determined to be *serially dependent*, then the VaR model needs to be revised to incorporate the correlations that are evident in the current conditions.

Professor's Note: For the exam, you do not need to know how to calculate the log-likelihood test statistic for conditional coverage. Therefore, the focus here is to understand that the test for conditional coverage should be performed when exceptions are clustered together.

Basel Committee Rules for Backtesting

LO 3.6: Describe the Basel rules for backtesting.

In the backtesting process, we attempt to strike a balance between the probability of a Type I error (rejecting a model that is correct) and a Type II error (failing to reject a model that is incorrect). Thus, the Basel Committee is primarily concerned with identifying whether exceptions are the result of bad luck (Type I error) or a faulty model (Type II error). The Basel Committee requires that market VaR be calculated at the 99% confidence level and backtested over the past year. At the 99% confidence level, we would expect to have 2.5 exceptions (250 × 0.01) each year, given approximately 250 trading days.

Regulators do not have access to every parameter input of the model and must construct rules that are applicable across institutions. To mitigate the risk that banks willingly commit a Type II error and use a faulty model, the Basel Committee designed the **Basel penalty zones** presented in Figure 5. The committee established a scale of the number of exceptions and corresponding increases in the capital multiplier, *k*. Thus, banks are penalized for exceeding four exceptions per year. The multiplier is normally three but can be increased to as much as four, based on the accuracy of the bank's VaR model. Increasing *k* significantly increases the amount of capital a bank must hold and lowers the bank's performance measures, like return on equity.

Notice in Figure 5 that there are three zones. The green zone is an acceptable number of exceptions. The yellow zone indicates a penalty zone where the capital multiplier is increased by 0.40 to 1.00. The red zone, where 10 or more exceptions are observed, indicates the strictest penalty with an increase of 1 to the capital multiplier.

Figure 5: Basel Penalty Zones

Zone	*Number of Exceptions*	*Multiplier (k)*
Green	0 to 4	3.00
Yellow	5	3.40
	6	3.50
	7	3.65
	8	3.75
	9	3.85
Red	10 or more	4.00

As shown in Figure 5, the yellow zone is quite broad (five to nine exceptions). The penalty (raising the multiplier from three to four) is automatically required for banks with 10 or more exceptions. However, the penalty for banks with five to nine exceptions is subject to supervisors' discretions, based on what type of model error caused the exceptions. The Committee established four categories of causes for exceptions and guidance for supervisors for each category:

- *The basic integrity of the model is lacking.* Exceptions occurred because of incorrect data or errors in the model programming. The penalty should apply.
- *Model accuracy needs improvement.* The exceptions occurred because the model does not accurately describe risks. The penalty should apply.
- *Intraday trading activity.* The exceptions occurred due to trading activity (VaR is based on static portfolios). The penalty should be *considered.*
- *Bad luck.* The exceptions occurred because market conditions (volatility and correlations among financial instruments) significantly varied from an accepted norm. These exceptions should be expected to occur at least some of the time. No penalty guidance is provided.

Although the yellow zone is broad, an accurate model could produce five or more exceptions 10.8% of the time at the 99% confidence level. So even if a bank has an accurate model, it is subject to punishment 10.8% of the time (using the required 99% confidence level). However, regulators are more concerned about Type II errors, and the increased capital multiplier penalty is enforced using the 97% confidence level. At this level, inaccurate models would not be rejected 12.8% of the time (e.g., those with VaR calculated at the 97% confidence level rather than the required 99% confidence level). While this seems to be only a slight difference, using a 99% confidence level would result in a 1.24 times greater level of required capital, providing a powerful economic incentive for banks to use a lower confidence level. Exemptions may be excluded if they are the result of bad luck that follows from an unexpected change in interest rates, exchange rates, political event, or natural disaster. Bank regulators keep the description of exceptions intentionally vague to allow adjustments during major market disruptions.

Industry analysts have suggested lowering the required VaR confidence level to 95% and compensating by using a greater multiplier. This would result in a greater number of expected exceptions, and variances would be more statistically significant. The one-year exception rate at the 95% level would be 13, and with more than 17 exceptions, the probability of a Type I error would be 12.5% (close to the 10.8% previously noted), but the probability of a Type II error at this level would fall to 7.4% (compared to 12.8% at a 97.5% confidence level). Thus, inaccurate models would fail to be rejected less frequently.

Another way to make variations in the number of exceptions more significant would be to use a longer backtesting period. This approach may not be as practical because the nature of markets, portfolios, and risk changes over time.

Key Concepts

LO 3.1

Backtesting is an important part of VaR model validation. It involves comparing the number of instances where the actual loss exceeds the VaR level (called exceptions) with the number predicted by the model at the chosen level of confidence. The Basel Committee requires banks to backtest internal VaR models and penalizes banks with excessive exceptions in the form of higher capital requirements.

LO 3.2

VaR models are based on static portfolios, while actual portfolio compositions are dynamic and incorporate fees, commissions, and other profit and loss factors. This effect is minimized by backtesting with a relatively short time horizon such as daily holding periods. The backtesting period constitutes a limited sample, and a challenge for risk managers is to find an acceptable level of exceptions.

LO 3.3

The failure rate of a model backtest is the number of exceptions divided by the number of observations: N / T. The Basel Committee requires backtesting at the 99% confidence level over the past year (250 business days). At this level, we would expect 250×0.01, or 2.5 exceptions.

LO 3.4

In using backtesting to accept or reject a VaR model, we must balance the probabilities of two types of errors: a Type I error is rejecting an accurate model, and a Type II error is failing to reject an inaccurate model. A log-likelihood ratio is used as a test for the validity of VaR models.

LO 3.5

Unconditional coverage testing does not evaluate the timing of exceptions, while conditional coverage tests review the number and timing of exceptions for independence. Current market or trading portfolio conditions may require changes to the VaR model.

LO 3.6

The Basel Committee penalizes financial institutions when the number of exceptions exceeds four. The corresponding penalties incrementally increase the capital requirement multiplier for the financial institution from three to four as the number of exceptions increase.

Concept Checkers

1. In backtesting a value at risk (VaR) model that was constructed using a 97.5% confidence level over a 252-day period, how many exceptions are forecasted?
   A. 2.5.
   B. 3.7.
   C. 6.3.
   D. 12.6.

2. Unconditional testing does not reflect the:
   A. size of the portfolio.
   B. number of exceptions.
   C. confidence level chosen.
   D. timing of the exceptions.

3. Which of the following statements regarding verification of a VaR model by examining its failure rates is false?
   A. The frequency of exceptions should correspond to the confidence level used for the model.
   B. According to Kupiec (1995), we should reject the hypothesis that the model is correct if the log-likelihood ratio (LR) > 3.84.
   C. Backtesting VaR models with a higher probability of exceptions is difficult because the number of exceptions is not high enough to provide meaningful information.
   D. The range for the number of exceptions must strike a balance between the chances of rejecting an accurate model (a Type I error) and the chances of failing to reject an inaccurate model (a Type II error).

4. The Basel Committee has established four categories of causes for exceptions. Which of the following does not apply to one of those categories?
   A. The sample is small.
   B. Intraday trading activity.
   C. Model accuracy needs improvement.
   D. The basic integrity of the model is lacking.

5. A risk manager is backtesting a sample at the 95% confidence level to see if a VaR model needs to be recalibrated. He is using 252 daily returns for the sample and discovered 17 exceptions. What is the *z*-score for this sample when conducting VaR model verification?
   A. 0.62.
   B. 1.27.
   C. 1.64.
   D 2.86.

Concept Checker Answers

1. **C** $(1 - 0.975) \times 252 = 6.3$

2. **D** Unconditional testing does not capture the timing of exceptions.

3. **C** Backtesting VaR models with a *lower probability of exceptions* is difficult because the number of exceptions is not high enough to provide meaningful information.

4. **A** Causes include the following: bad luck, intraday trading activity, model accuracy needs improvement, and the basic integrity of the model is lacking.

5. **B** The *z*-score is calculated using x = 17, p = 0.05, c = 0.95, and N = 252, as follows:

$$z = \frac{17 - 0.05(252)}{\sqrt{0.05(0.95)252}} = \frac{17 - 12.6}{\sqrt{11.97}} = \frac{4.4}{3.4598} = 1.27$$

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

VaR Mapping

Topic 4

Exam Focus

This topic introduces the concept of mapping a portfolio and shows how the risk of a complex, multi-asset portfolio can be separated into risk factors. For the exam, be able to explain the mapping process for several types of portfolios, including fixed-income portfolios and portfolios consisting of linear and nonlinear derivatives. Also, be able to describe how the mapping process simplifies risk management for large portfolios. Finally, be able to distinguish between general and specific risk factors, and understand the various inputs required for calculating undiversified and diversified value at risk (VaR).

The Mapping Process

LO 4.1: Explain the principles underlying VaR mapping, and describe the mapping process.

Value at risk (VaR) mapping involves replacing the current values of a portfolio with risk factor exposures. The first step in the process is to measure all current positions within a portfolio. These positions are then mapped to **risk factors** by means of **factor exposures**. Mapping involves finding common risk factors among positions in a given portfolio. If we have a portfolio consisting of a large number of positions, it may be difficult and time consuming to manage the risk of each individual position. Instead, we can evaluate the value of these positions by mapping them onto common risk factors (e.g., changes in interest rates or equity prices). By reducing the number of variables under consideration, we greatly simplify the risk management process.

Mapping can assist a risk manager in evaluating positions whose characteristics may change over time, such as fixed-income securities. Mapping can also provide an effective way to manage risk when there is not sufficient historical data for an investment, such as an initial public offering (IPO). In both cases, evaluating historical prices may not be relevant, so the manager must evaluate those risk factors that are likely to impact the portfolio's risk profile.

The principles for VaR risk mapping are summarized as follows:

- VaR mapping aggregates risk exposure when it is impractical to consider each position separately. For example, there may be too many computations needed to measure the risk for each individual position.
- VaR mapping simplifies risk exposures into primitive risk factors. For example, a portfolio may have thousands of positions linked to a specific exchange rate that could be summarized with one aggregate risk factor.
- VaR risk measurements can differ from pricing methods where prices cannot be aggregated. The aggregation of a number of positions to one risk factor is acceptable for risk measurement purposes.

- VaR mapping is useful for measuring changes over time, as with bonds or options. For example, as bonds mature, risk exposure can be mapped to spot yields that reflect the current position.
- VaR mapping is useful when historical data is not available.

The first step in the VaR mapping process is to identify common risk factors for different investment positions. Figure 1 illustrates how the market values (MVs) of each position or investment are matched to the common risk factors identified by a risk manager.

Figure 1: Mapping Positions to Risk Factors

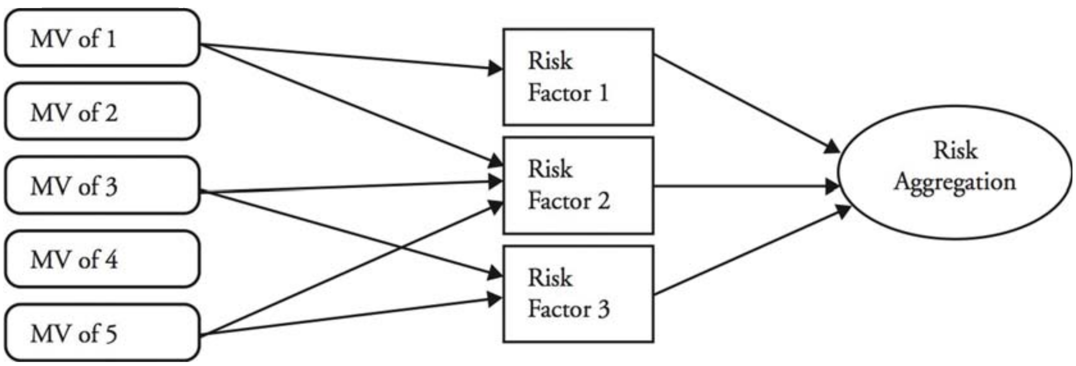


Figure 2 illustrates the next step, where the risk manager constructs risk factor distributions and inputs all data into the **risk model**. In this case, the market value of the first position, MV_1, is allocated to the risk exposures in the first row, x_{11}, x_{12}, and x_{13}. The other market value positions are linked to the risk exposures in a similar way. Summing the risk factors in each column then creates a vector consisting of three risk exposures.

Figure 2: Mapping Risk Exposures

Investment	*Market Value*	*Risk Factor 1*	*Risk Factor 2*	*Risk Factor 3*
1	MV_1	x_{11}	x_{12}	x_{13}
2	MV_2	x_{21}	x_{22}	x_{23}
3	MV_3	x_{31}	x_{32}	x_{33}
4	MV_4	x_{41}	x_{42}	x_{43}
5	MV_5	x_{51}	x_{52}	x_{53}

LO 4.2: Explain how the mapping process captures general and specific risks.

So how many **general risk factors** (or **primitive risk factors**) are appropriate for a given portfolio? In some cases, one or two risk factors may be sufficient. Of course, the more risk factors chosen, the more time consuming the modeling of a portfolio becomes. However, more risk factors could lead to a better approximation of the portfolio's risk exposure.

In our choice of general risk factors for use in VaR models, we should be aware that the types and number of risk factors we choose will have an effect on the size of residual or **specific risks**. **Specific risks** arise from unsystematic risk or asset-specific risks of various positions in the portfolio. The more precisely we define risk, the smaller the specific risk.

For example, a portfolio of bonds may include bonds of different ratings, terms, and currencies. If we use duration as our only risk factor, there will be a significant amount of variance among the bonds that we referred to as specific risk. If we add a risk factor for credit risk, we could expect that the amount of specific risk would be smaller. If we add another risk factor for currencies, we would expect that the specific risk would be even smaller. Thus, the definition of specific risk is a function of general market risk.

As an example, suppose an equity portfolio consists of 5,000 stocks. Each stock has a market risk component and a firm-specific component. If each stock has a corresponding risk factor, we would need roughly 12.5 million covariance terms (i.e., [5,000 × (5,000 – 1)] / 2) to evaluate the correlation between each risk factor. To simplify the number of parameters required, we need to understand that diversification will reduce firm-specific components and leave only market risk (i.e., systematic risk or beta risk). We can then map the market risk component of each stock onto a stock index (i.e., changes in equity prices) to greatly reduce the number of parameters needed.

Suppose you have a portfolio of *N* stocks and map each stock to the market index, which is defined as your primitive risk factor. The risk exposure, β_i, is computed by regressing the return of stock *i* on the market index return using the following equation:

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$$

We can ignore the first term (i.e., the intercept) as it does not relate to risk, and we will also assume that the last term, which is related to specific risk, is not correlated with other stocks or the market portfolio. If the weight of each position in the portfolio is defined as w_i, then the portfolio return is defined as follows:

$$R_P = \sum_{i=1}^{N} w_i R_i = \sum_{i=1}^{N} w_i \beta_i R_M + \sum_{i=1}^{N} w_i \varepsilon_i$$

Aggregating all risk exposures, β_i, based on the market weights of each position determines the risk exposure as follows:

$$\beta_P = \sum_{i=1}^{N} w_i \beta_i$$

We can then decompose the variance, *V*, of the portfolio return into two components, which consist of general market risk exposures and specific risk exposures, as follows:

$$V(R_P) = \beta_P^2 \times V(R_M) + \sum_{i=1}^{N} w_i^2 \times \sigma_{\varepsilon,i}^2$$

General market risk: $\beta_P^2 \times V(R_M)$

Specific risk: $\sum_{i=1}^{N} w_i^2 \times \sigma_{\varepsilon,i}^2$

Mapping Approaches for Fixed-Income Portfolios

LO 4.3: Differentiate among the three methods of mapping portfolios of fixed income securities.

After we have selected our general risk factors, we must map our portfolio onto these factors. The three methods of mapping for fixed-income securities are (1) principal mapping, (2) duration mapping, and (3) cash flow mapping.

Principal mapping. This method includes only the risk of repayment of principal amounts. For principal mapping, we consider the average maturity of the portfolio. VaR is calculated using the risk level from the zero-coupon bond that equals the average maturity of the portfolio. This method is the simplest of the three approaches.

Duration mapping. With this method, the risk of the bond is mapped to a zero-coupon bond of the same duration. For duration mapping, we calculate VaR by using the risk level of the zero-coupon bond that equals the duration of the portfolio. Note that it may be difficult to calculate the risk level that exactly matches the duration of the portfolio.

Cash flow mapping. With this method, the risk of the bond is decomposed into the risk of each of the bond's cash flows. Cash flow mapping is the most precise method because we map the present value of the cash flows (i.e., face amount discounted at the spot rate for a given maturity) onto the risk factors for zeros of the same maturities and include the inter-maturity correlations.

LO 4.4: Summarize how to map a fixed income portfolio into positions of standard instruments.

To illustrate principal, duration, and cash flow mapping, we will use a two position fixed-income portfolio consisting of a one-year bond and a five-year bond. You will notice in the following examples that the primary difference between these mapping techniques is the consideration of the timing and amount of cash flows.

Suppose a portfolio consists of two par value bonds. One bond is a one-year $100 million bond with a coupon rate of 3.5%. The second bond is a five-year $100 million bond with a coupon rate of 5%. In this example, we will differentiate between the timing and cash flows used to map the VaR for this portfolio using principal mapping, duration mapping, and cash flow mapping. The risk percentages (or VaR percentages) for zero-coupon bonds with maturities ranging from one to five years (at the 95% confidence level) are as follows:

Maturity	*VAR %*
1	0.4696
2	0.9868
3	1.4841
4	1.9714
5	2.4261

Principal mapping is the simplest of the three techniques as it only considers the timing of the redemption or maturity payments of the bonds. While this simplifies the process, it ignores all coupon payments for the bonds. The weights in this example are both 50% (i.e., \$100 million / \$200 million). Thus, the weighted average life of this portfolio for the two bonds is three years [0.50(1) + 0.50(5) = 3].

As Figure 3 illustrates, the principal mapping technique assumes that the total portfolio value of \$200 million occurs at the average life of the portfolio, which is three years. Note that the VaR percentage at the 95% confidence level is 1.4841 for a three-year zero-coupon bond. We compute the VaR under the principal method by multiplying the VaR percentage times the market value of the average life of the bond, as follows:

Principal mapping VaR = \$200 million × 1.4841% = \$2.968 million

Figure 3: Fixed-Income Mapping Techniques

				Mapping Technique		
Year	*CFs for 5-Year Bond*	*CFs for 1-Year Bond*	*Spot Rates*	*Principal*	*Duration*	*PV(CF)*
1	\$5	\$103.5	3.50%			\$104.83
2	\$5	\$0	3.90%			\$4.63
2.768					\$200	
3	\$5	\$0	4.19%	\$200		\$4.42
4	\$5	\$0	4.21%			\$4.24
5	\$105	\$0	5.10%			\$81.88
				\$200	\$200	\$200.00

In the last three columns of Figure 3, you can see the differences in the amounts and timing of cash flows for all three methods. To calculate the VaR of this fixed-income portfolio using duration mapping, we simply replace the portfolio with a zero-coupon bond that has the same maturity as the duration of the portfolio. Figure 4 demonstrates the calculation of Macaulay duration for this portfolio. The numerator of the duration calculation is the sum of time, *t*, multiplied by the present value of cash flows, and the denominator is simply the present value of all cash flows. Duration is then computed as \$553.69 million / \$200 million = 2.768.

Figure 4: Duration Calculation

Year	*CF for 5-Year Bond*	*CF for 1-Year Bond*	*Spot Rate*	*PV(CF)*	*t × PV(CF)*
1	\$5	\$103.5	3.50%	\$104.83	\$104.83
2	\$5	\$0	3.90%	\$4.63	\$9.26
3	\$5	\$0	4.19%	\$4.42	\$13.26
4	\$5	\$0	4.21%	\$4.24	\$16.96
5	\$105	\$0	5.10%	\$81.88	\$409.38
				\$200.00	\$553.69

The next step is to interpolate the VaR for a zero-coupon bond with a maturity of 2.768 years. Recall that the VaR percentages for two-year and three-year zero-coupon bonds were 0.9868 and 1.4841, respectively.

The VaR of a 2.768 year maturity zero-coupon bond is interpolated as follows:

$$0.9868 + (1.4841 - 0.9868) \times (2.768 - 2) = 0.9868 + (0.4973 \times 0.768) = 1.3687$$

We now have the information needed to calculate the VaR for this portfolio using the interpolated VaR percentage for a zero-coupon bond with a 2.768 year maturity:

Duration mapping VaR = \$200 million × 1.3687% = \$2.737 million

In order to calculate the VaR for this fixed-income portfolio using cash flow mapping, we need to map the present value of the cash flows (i.e., face amount discounted at the spot rate for a given maturity) onto the risk factors for zeros of the same maturities and include the inter-maturity correlations. Figure 5 summarizes the required calculations. The second column of Figure 5 provides the present value of cash flows that were computed in Figure 3. The third column of Figure 5 multiplies the present value of cash flows times the zero-coupon VaR percentages.

Figure 5: Cash Flow Mapping

			Correlation Matrix (R)					
Year	*x*	*x × V*	*1Y*	*2Y*	*3Y*	*4Y*	*5Y*	*xΔVaR*
1	104.83	0.4923	1	0.894	0.887	0.871	0.861	1.17
2	4.63	0.0457	0.894	1	0.99	0.964	0.954	0.115
3	4.42	0.0656	0.887	0.99	1	0.992	0.987	0.170
4	4.24	0.0836	0.871	0.964	0.992	1	0.996	0.217
5	81.88	1.9864	0.861	0.954	0.987	0.996	1	5.168
Undiversified VaR		2.674						6.840
							Diversified VaR	2.615

If the five zero-coupon bonds were all perfectly correlated, then the **undiversified VaR** could be calculated as follows:

$$\text{Undiversified VaR} = \sum_{i=1}^{N} |x_i| \times V_i$$

In this example, the undiversified VaR is computed as the sum of the third column: 2.674.

The correlation matrix provided in the fourth through eighth columns of Figure 5 provides the inter-maturity correlations for the zero-coupon bonds for all five maturities. The **diversified VaR** can be computed using matrix algebra as follows:

$$\text{Diversified VaR} = \alpha\sqrt{x'\sum x} = \sqrt{(x \times V)'R(x \times V)}$$

Where *x* is the present value of cash flows vector, *V* is the vector of VaR for zero-coupon bond returns and *R* is the correlation matrix. The last column of Figure 5 summarizes the computations for the matrix algebra. The square root of the sum of this column (6.840) is the diversified VaR using cash flow mapping and is calculated as 2.615.

Notice that in order to calculate portfolio diversified VaR using the cash flow mapping method, we need to incorporate the correlations between the zero-coupon bonds. As you can see, cash flow mapping is the most precise method, but it is also the most complex.

Professor's Note: The complex calculations required for cash flow mapping would be very time consuming to perform using a financial calculator. Therefore, this calculation it is highly unlikely to show up on the exam.

Stress Testing

LO 4.5: Describe how mapping of risk factors can support stress testing.

If we assume that there is perfect correlation among maturities of the zeros, the portfolio VaR would be equal to the **undiversified VaR** (i.e., the sum of the VaRs, as illustrated in column 3 of Figure 5). Instead of calculating the undiversified VaR directly, we could reduce each zero-coupon value by its respective VaR and then revalue the portfolio. The difference between the revalued portfolio and the original portfolio value should be equal to the undiversified VaR. Stressing each zero by its VaR is a simpler approach than incorporating correlations; however, this method ceases to be viable if correlations are anything but perfect (i.e., 1).

Using the same two-bond portfolio from the previous example, we can stress test the VaR measurement, assuming all zeros are perfectly correlated, and derive movements in the value of zero-coupon bonds. Figure 6 illustrates the calculations required to stress test the portfolio. The present value factor for a one-year zero-coupon bond discounted at 3.5% is simply 1 / (1.035) = 0.9662. The VaR percentage movement at the 95% confidence level for a one-year zero-coupon bond is provided in column 5 (0.4696). Thus, there is a 95% probability that a one-year zero-coupon bond will fall to 0.9616 [computed as follows: 0.9662 × (1 – 0.4696 / 100) = 0.9616].

The VaR adjusted present values of zero-coupon bonds are presented in column 7 of Figure 6. The last column simply finds the present value of the portfolio's cash flows using the VaR% adjusted present value factors. The sum of these values suggests that the change in portfolio value is $2.67 (computed $200.00 – $197.33). Notice that the $2.67 is equivalent to the undiversified VaR previously computed in Figure 5.

Figure 6: Stress Testing a Portfolio

Year	*Portfolio CF*	*Spot Rate*	*PV(CF)*	*VAR %*	*PV Factor*	*VaR Adj. PV Factor*	*New Zero Value*
1	$108.5	3.50%	$104.83	0.4696	0.9662	0.9616	$104.34
2	$5	3.90%	$4.63	0.9868	0.9263	0.9172	$4.59
3	$5	4.19%	$4.42	1.4841	0.8841	0.8710	$4.36
4	$5	4.21%	$4.24	1.9714	0.8479	0.8312	$4.16
5	$105	5.10%	$81.88	2.4261	0.7798	0.7609	$79.89
			$200.00				$197.33

Benchmarking a Portfolio

LO 4.6: Explain how VaR can be used as a performance benchmark.

It is often convenient to measure VaR relative to a benchmark portfolio. This is what is referred to as benchmarking a portfolio. Portfolios can be constructed that match the risk factors of a benchmark portfolio but have either a higher or a lower VaR. The VaR of the deviation between the two portfolios is referred to as a **tracking error VaR**. In other words, tracking error VaR is a measure of the difference between the VaR of the target portfolio and the benchmark portfolio.

Suppose you are trying to benchmark the VaR of a $100 million bond portfolio with a duration of 4.77 to a portfolio of two zero-coupon bonds with the same duration at the 95% confidence level. The market value weights of the bonds in the benchmark portfolio and portfolios of two zero-coupon bonds are provided in Figure 7.

Figure 7: Benchmark Portfolio and Zero-Coupon Bond Portfolio Weights

Maturity	*Benchmark*	*A*	*B*	*C*	*D*	*E*
1 month	1.00					84.35
3 month	1.25					
6 month	2.00					
1 year	12.50				58.10	
2 year	23.50			60.50		
3 year	17.50		55.75			
4 year	12.00	23.00				
5 year	8.00	77.00				
7 year	6.50		44.25			
9 year	4.50			39.50		
10 year	3.50				41.90	
15 year	3.00					
20 year	3.25					
30 year	1.50					15.65
Total Value	100.00	100.00	100.00	100.00	100.00	100.00

The first step in the benchmarking process is to match the duration with two zero-coupon bonds. Therefore, the weights of the market values of the zero-coupon bonds in Figure 7 are adjusted to match the benchmark portfolio duration of 4.77. Figure 8 illustrates the creation of five two-bond portfolios with a duration of 4.77. The market values of all bonds in the zero-coupon portfolios are adjusted to match the duration of the benchmark portfolio. For example, portfolio A in Figures 7 and 8 is comprised of a four-year zero-coupon bond with a market weight of 23% and a five-year zero-coupon bond with a market weight of 77%. This results in a duration for portfolio A of 4.77, which is equivalent to the benchmark. The other zero-coupon bond portfolios also adjust their weights of the two zero-coupon bonds to match the benchmark's duration.

Figure 8: Matching Duration of Zero-Coupon Bond Portfolios to Benchmark

Time	*Benchmark*	*A*	*B*	*C*	*D*	*E*
1 month	0.00					0.07
3 month	0.00					
6 month	0.01					
1 year	0.13				0.58	
2 year	0.47			1.21		
3 year	0.53		1.67			
4 year	0.48	0.92				
5 year	0.40	3.85				
7 year	0.46		3.10			
9 year	0.41			3.56		
10 year	0.35				4.19	
15 year	0.45					
20 year	0.65					
30 year	0.45					4.70
Duration	4.77	4.77	4.77	4.77	4.77	4.77

Figure 9 presents the absolute VaR by multiplying the market value weights of the bonds (presented in Figure 7) by the VaR percentages presented in column 2 of Figure 9. The VaR percentages are for a monthly time horizon. The absolute VaR for the benchmark portfolio is computed as $1.99 million. Notice this is very close to the VaR percentage for the four-year note in Figure 9.

Next, the absolute VaR for the five portfolios each consisting of two zero-coupon bonds is computed by multiplying the VaR percentage times the market value of the zero-coupon bonds. We define the new vector of market value positions for each zero-coupon bond portfolio presented in Figure 7 as x and the vector of market value positions of the benchmark as x_0. Then the relative performance to the benchmark is computed as the tracking error (TE) VaR as follows:

$$\text{Tracking error VaR} = \alpha\sqrt{(x - x_0)'\sum(x - x_0)}$$

The tracking error or difference between the VaR for the benchmark and zero-bond portfolios is due to nonparallel shifts in the term structure of interest rates. However, the tracking error of \$0.45 million for zero-coupon bond portfolio A and the benchmark is much less than the VaR for the benchmark at \$1.99. In this example, the smallest tracking error is for portfolio C. Notice that the benchmark portfolio has the largest market weight in the two-year note. Thus, the cash flows are most closely aligned with portfolio C, which contains a two-year zero-coupon bond. This reduces the tracking error to \$0.17 million for that portfolio. Also notice that minimizing the absolute VaR in Figure 9 is not the same as minimizing the tracking error. Portfolio E is a barbell portfolio with the highest tracking error to the index, even though it has the lowest absolute VaR.

Tracking error can be used to compute the variance reduction (similar to R-squared in a regression) as follows:

$$\text{Variance improvement} = 1 - (\text{tracking error} / \text{benchmark VaR})^2$$

Variance improvement for portfolio C relative to the benchmark is computed as:

$$1 - (0.17 / 1.99)^2 = 99.3\%$$

Figure 9: Absolute VaR and Tracking Error Relative to Benchmark Portfolio

Time	*VaR%*	*Benchmark*	*A*	*B*	*C*	*D*	*E*
1 month	0.022	0.00					0.02
3 month	0.065	0.00					
6 month	0.163	0.00					
1 year	0.47	0.06				0.27	
2 year	0.987	0.23			0.60		
3 year	1.484	0.26		0.83			
4 year	1.971	0.24	0.45				
5 year	2.426	0.19	1.87				
7 year	3.192	0.21		1.41			
9 year	3.913	0.18			1.55		
10 year	4.25	0.15				1.78	
15 year	6.234	0.19					
20 year	8.146	0.26					
30 year	11.119	0.17					1.74
Absolute VaR		**1.99**	**2.32**	**2.24**	**2.14**	**2.05**	**1.76**
Tracking Error VaR		**0.00**	**0.45**	**0.31**	**0.17**	**0.21**	**0.84**

LO 4.7: Describe the method of mapping forwards, forward rate agreements, interest rate swaps, and options.

Mapping Approaches for Linear Derivatives

Forward Contracts

The delta-normal method provides accurate estimates of VaR for portfolios and assets that can be expressed as linear combinations of normally distributed risk factors. Once a portfolio, or financial instrument, is expressed as a linear combination of risk factors, a covariance (correlation) matrix can be generated, and VaR can be measured using matrix multiplication.

Forwards are appropriate for the application of the delta-normal method. Their values are a linear combination of a few general risk factors, which have commonly available volatility and correlation data.

The current value of a forward contract is equal to the present value of the difference between the current forward rate, F_t, and the locked in delivery rate, K, as follows:

$$\text{Forward}_t = (F_t - K)e^{-rt}$$

Suppose you wish to compute the diversified VaR of a forward contract that is used to purchase euros with U.S. dollars one year from now. This forward position is analogous to the following three separate risk positions:

1. A short position in a U.S. Treasury bill.
2. A long position in a one-year euro bill.
3. A long position in the euro spot market.

Figure 10 presents the pricing information for the purchase of $100 million euros in exchange for $126.5 million, as well as the correlation matrix between the positions.

Figure 10: Monthly VaR for Forward Contract and Correlation Matrix

Risk Factor	*Price/Rate*	*VaR%*	*EUR Spot*	*1Yr EUR*	*1Yr US*
EUR spot	1.2500	4.5381	1.000	0.115	0.073
Long EUR bill	0.0170	0.1396	0.115	1.000	–0.047
Short USD bill	0.0292	0.2121	0.073	–0.047	1.000
EUR forward	1.2650				

In this example, we have a long position in a EUR contract worth $122.911 million today and a short position in a one-year U.S. T-bill worth $122.911 today, as illustrated in Figure 11. The fourth column represents the investment present values. The fifth column represents the absolute present value of cash flows multiplied by the VaR percentage from Figure 10.

Figure 11: Undiversified and Diversified VaR of Forward Contract

Position	*PV factor*	*CF*	*x*	$\lvert x_i \rvert V_i$	*xΔVaR*
EUR spot			122.911	5.578	31.116
Long EUR bill	0.9777	100.0	122.911	0.172	0.142
Short USD bill	0.9678	126.5	122.911	0.261	–0.036
Undiversified VaR				**6.010**	31.221
Diversified VaR					**5.588**

*Note that some rounding has occurred.

The undiversified VaR for this position is $6.01 million, and the diversified VaR for this position is $5.588 million. Recall that the diversified VaR is computed using matrix algebra.

The general procedure we've outlined for forwards also applies to other types of financial instruments, such as forward rate agreements and interest rate swaps. As long as an instrument can be expressed as linear combinations of its basic components, the delta-normal VaR may be applied with reasonable accuracy.

Forward Rate Agreements (FRA)

Suppose you have an FRA that locks in an interest rate one year from now. Figure 12 illustrates data related to selling a 6 × 12 FRA on $100 million. This amount is equivalent to borrowing $100 million for a 6-month period (180 days) and investing the proceeds at the 12-month rate (360 days). Assuming that the 360-day spot rate is 4.5% and the 180-day spot rate is 4.1%, the present values of the cash flows are presented in the second column of Figure 12. The present value of the notional $100 million contract is x = $100 / 1.0205 = $97.991 million. This will be invested for a 12-month period. The forward rate is then computed as follows: $(1 + F_{1,2} / 2) = [1.045 / (1 + 0.041 / 2)] = [(1.045 / 1.0205) - 1] \times 2 = 4.8\%$.

The sixth column computes the undiversified VaR of $0.62 million at the 95% confidence level using the VaR percentages in the third column multiplied by the absolute value of the present values of cash flows. Matrix algebra is then used to multiply this vector by the correlation matrix presented in columns four and five to compute the diversified VaR of $0.348 million.

Figure 12: Calculating VaR for an FRA

Position	*PV(CF), x*	*VaR%*	*Correlations (R)*		$\lvert x_i \rvert V_i$	*xΔVaR*
180 days	–97.991	0.1629	1	0.79	0.160	–0.0325
360 days	97.991	0.4696	0.79	1	0.460	0.1537
Undiversified VaR					**0.620**	
						0.1212
Diversified VaR						**0.348**

Interest Rate Swaps

Interest rate swaps are commonly used to exchange interest rates from fixed to floating rates or from floating to fixed rates. Thus, an interest rate swap can be broken down into fixed and floating parts. The fixed part is priced with a coupon-paying bond and the floating part is priced as a floating-rate note.

Suppose you want to compute the VaR of a $100 million four-year swap that pays a fixed rate for four years in exchange for a floating-rate payment. The necessary steps to compute the undiversified and diversified VaR amounts are as follows:

Step 1: Begin by creating a present value of cash flows showing the short position of the fixed portion as we agree to pay the fixed interest rates and fixed bond maturity. Then, add the long present value of the variable rate bond at a present value of $100 million today.

Step 2: Multiply the vector representing the absolute present values of cash flows by the VaR percentages at the 95% confidence level and sum the values to compute the undiversified VaR amount.

Step 3: Use matrix algebra to multiply the correlation matrix by the absolute values to compute the diversified VaR amount. Again, recall that the diversified VaR is computed using matrix algebra.

Mapping Approaches for Nonlinear Derivatives

As mentioned, the delta-normal VaR method is based on linear relationships between variables. Options, however, exhibit nonlinear relationships between movements of the values of the underlying instruments and the values of the options. In many cases, the delta-normal method may still be applied because the value of an option may be expressed linearly as the product of the option delta and the underlying asset.

Unfortunately, the delta-normal VaR cannot be expected to provide an accurate estimate of the true VaR over ranges where deltas are unstable. In other words, over longer periods of time, the delta is not a constant, which makes linear methods inappropriate. Conversely, over short periods of time, such as one day, a linear approximation of the delta is more accurate. However, the accuracy of this approximation is dependent on parameter inputs (i.e., delta increases with the underlying spot price).

For example, assume the strike price of an option is $100 with a volatility of 25%. If we are only concerned about a one-day risk horizon, then the one-day loss could be computed as follows:

$$-\alpha S\sigma\sqrt{T} = -1.645 \times \$100 \times 0.25 \times \sqrt{\frac{1}{252}} = -\$2.59$$

Thus, over a one-day horizon, the worst case scenario at the 95% confidence level is a loss of $2.59, which brings the position down to $97.41. Linear approximations using this method may be reliable for longer maturity options if the risk horizon is very short, such as a one-day time horizon.

Professor's Note: Options are usually mapped using a Taylor series approximation and using the delta-gamma method to calculate the option VaR.

Key Concepts

LO 4.1

Value at risk (VaR) mapping involves replacing the current values of a portfolio with risk factor exposures. Portfolio exposures are broken down into general risk factors and mapped onto those factors.

LO 4.2

Specific risk decreases as more risk factors are added to a VaR model.

LO 4.3

Fixed-income risk mapping methods include principal mapping, duration mapping, and cash flow mapping. Principal mapping considers only the principal cash flow at the average life of the portfolio. Duration mapping considers the market value of the portfolio at its duration. Cash flow mapping is the most complex method considering the timing and correlations of all cash flows.

LO 4.4

The primary difference between principal, duration, and cash flow mapping techniques is the consideration of the timing and amount of cash flows.

Undiversified VaR is calculated as:

$$\text{Undiversified VaR} = \sum_{i=1}^{N} |x_i| \times V_i$$

Diversified VaR is computed using matrix algebra as follows:

$$\text{Diversified VaR} = \alpha\sqrt{x'\sum x} = \sqrt{(x \times V)'R(x \times V)}$$

LO 4.5

Stress testing each zero-coupon bond by its VaR is a simpler approach than incorporating correlations; however, this method ceases to be viable if correlations are anything other than 1.

LO 4.6

A popular use of VaR is to establish a benchmark portfolio and measure VaR of other portfolios in relation to this benchmark. The tracking error VaR is smallest for portfolios most closely matched based on cash flows.

LO 4.7

Delta-normal VaR can be applied to portfolios of many types of instruments as long as the risk factors are linearly related. Application of the delta-normal method with options and other derivatives does not provide accurate VaR measures over long risk horizons in which deltas are unstable.

Concept Checkers

1. Which of the following methods is not one of the three approaches for mapping a portfolio of fixed-income securities onto risk factors?
   A. Principal mapping.
   B. Duration mapping.
   C. Cash flow mapping.
   D. Present value mapping.

2. If portfolio assets are perfectly correlated, portfolio VaR will equal:
   A. marginal VaR.
   B. component VaR.
   C. undiversified VaR.
   D. diversified VaR.

3. Which of the following could be considered a general risk factor?
   I. Exchange rates.
   II. Zero-coupon bonds.

   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.

4. The VaR percentages at the 95% confidence level for a bond with maturities ranging from one year to five years are as follows:

Maturity	*VAR %*
1	0.4696
2	0.9868
3	1.4841
4	1.9714
5	2.4261

   A bond portfolio consists of a \$100 million bond maturing in two years and a \$100 million bond maturing in four years. What is the VaR of this bond portfolio using the principal VaR mapping method?
   A. \$1.484 million.
   B. \$1.974 million.
   C. \$2.769 million.
   D. \$2.968 million.

5. Suppose you are calculating the tracking error VaR for two zero-coupon bonds using a $100 million benchmark bond portfolio with the following maturities and market value weights. Which of the following combinations of two zero-coupon bonds would most likely have the smallest tracking error?

Maturity	*Benchmark*
1 month	1.00
1 year	10.00
2 year	13.00
3 year	24.00
4 year	12.00
5 year	18.00
7 year	9.25
10 year	6.50
20 year	4.75
30 year	1.50

A. 1 year and 7 year.
B. 2 year and 4 year.
C. 3 year and 5 year.
D. 4 year and 7 year.

Concept Checker Answers

1. **D** Present value mapping is not one of the approaches.

2. **C** If we assume perfect correlation among assets, VaR would be equal to undiversified VaR.

3. **A** Exchange rates can be used as general risk factors. Zero-coupon bonds are used to map bond positions but are not considered a risk factor. However, the interest rate on those zeros is a risk factor.

4. **D** The VaR percentage is 1.4841 for a three-year zero-coupon bond [(2 + 4) / 2 = 3]. We compute the VaR under the principal method by multiplying the VaR percentage times the market value of the average life of the bond: principal mapping VaR = \$200 million × 1.4841% = \$2.968 million.

5. **C** The three-year and five-year cash flows are highest for the benchmark portfolio at \$24 million and \$18 million, respectively. Thus, tracking error VaR will likely be the lowest for the portfolio where the cash flows of the benchmark and zero-coupon bond portfolios are most closely matched.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Messages from the Academic Literature on Risk Measurement for the Trading Book

Topic 5

Exam Focus

This topic addresses tools for risk measurement, including value at risk (VaR) and expected shortfall. Specifically, we will examine VaR implementation over different time horizons and VaR adjustments for liquidity costs. This topic also examines academic studies related to integrated risk management and discusses the importance of measuring interactions among risks due to risk diversification. Note that several concepts in this topic, such as liquidity risk, stressed VaR, and capital requirements, will be discussed in more detail in Book 3, which covers operational and integrated risk management and the Basel Accords.

Value at Risk (VaR) Implementation

LO 5.1: Explain the following lessons on VaR implementation: time horizon over which VaR is estimated, the recognition of time varying volatility in VaR risk factors, and VaR backtesting.

There is no consensus regarding the proper time horizon for risk measurement. The appropriate time horizon depends on the risk measurement purpose (e.g., setting capital limits) as well as portfolio liquidity. Thus, there is not a universally accepted approach for aggregating various VaR measures based on different time horizons.

Time-varying volatility results from volatility fluctuations over time. The effect of time-varying volatility on the accuracy of VaR measures decreases as time horizon increases. However, volatility generated by stochastic (i.e., random) jumps will reduce the accuracy of long-term VaR measures unless there is an adjustment made for stochastic jumps. It is important to recognize time-varying volatility in VaR measures since ignoring it will likely lead to an underestimation of risk. In addition to volatility fluctuations, risk managers should also account for time-varying correlations when making VaR calculations.

To simplify VaR estimation, the financial industry has a tendency to use short time horizons. This approach is computationally attractive for larger portfolios. However, a 10-day VaR time horizon, as suggested by the Basel Committee on Banking Supervision, is not always optimal. It is more preferred to instead allow the risk horizon to vary based on specific investment characteristics. When computing VaR over longer time horizons, a risk manager needs to account for the variation in a portfolio's composition over time. Thus, a longer than 10-day time horizon may be necessary for economic capital purposes.

Historically, VaR backtesting has been used to validate VaR models. However, backtesting is not effective when the number of VaR exceptions is small. In addition, backtesting is less effective over longer time horizons due to portfolio instability. VaR models tend to be more realistic if time-varying volatility is incorporated; however, this approach tends to generate a procyclical VaR measure and produces unstable risk models due to estimation issues.

Integrating Liquidity Risk into VaR Models

LO 5.2: Describe exogenous and endogenous liquidity risk and explain how they might be integrated into VaR models.

During times of a financial crisis, market liquidity conditions change, which changes the liquidity horizon of an investment (i.e., the time to unwind a position without materially affecting its price). Two types of liquidity risk are exogenous liquidity and endogenous liquidity. Both types of liquidity are important to measure; however, academic studies suggest that risk valuation models should first account for the impact of endogenous liquidity.

Professor's Note: In Book 3, we will examine the estimation of liquidity risk using the exogenous spread approach and the endogenous price approach.

Exogenous liquidity is handled through the calculation of a liquidity-adjusted VaR (LVaR) measure, and represents market-specific, average transaction costs. The LVaR measure incorporates a bid/ask spread by adding liquidity costs to the initial estimate of VaR.

Endogenous liquidity is an adjustment for the price effect of liquidating positions. It depends on trade sizes and is applicable when market orders are large enough to move prices. Endogenous liquidity is the elasticity of prices to trading volumes and is more easily observed in instances of high liquidity risk.

Poor market conditions can cause a "flight to quality," which decreases a trader's ability to unwind positions in thinly traded assets. Thus, endogenous liquidity risk is most applicable to exotic/complex trading positions and very relevant in high-stress market conditions, however, endogenous liquidity costs will be present in all market conditions.

Risk Measures

LO 5.3: Compare VaR, expected shortfall, and other relevant risk measures.

VaR estimates the maximum loss that can occur given a specified level of confidence. VaR is a useful measure of risk since it is easy to compute and readily applicable. However, it does not consider losses beyond the VaR confidence level (i.e., the threshold level). In other words, VaR does not consider the severity of losses in the tail of the returns distribution. An additional disadvantage of VaR is that it is not subadditive, meaning that the VaR of a combined portfolio can be greater than the sum of the VaRs of each asset within the portfolio.

An alternative risk measure, frequently used by financial institutions, is **expected shortfall**. Expected shortfall is more complex and computationally intensive than VaR, however, it does correct for some of the drawbacks of VaR. Namely, it is able to account for the magnitude of losses beyond the VaR threshold and it is always subadditive. In addition, the application of expected shortfall will mitigate the impact that a specific confidence level choice will have on risk management decisions.

Spectral risk measures generalize expected shortfall and consider an investment manager's aversion to risk. These measures have select advantages over expected shortfall by including better smoothness properties when weighting observations as well as the ability to modify a risk measure to reflect an investor's specific risk aversion. Aside from the special case of expected shortfall, other spectral risk measures are rarely used in practice.

Stress Testing

It is important to incorporate stress testing into risk models by selecting various stress scenarios. Three primary applications of stress testing exercises are as follows:

1. **Historical scenarios**, which examine previous market data.

2. **Predefined scenarios**, which attempt to assess the impact on profit/loss of adverse changes in a predetermined set of risk factors.

3. **Mechanical-search stress tests**, which use automated routines to cover possible changes in risk factors.

In stress testing, it is important to "stress" the correlation matrix. However, an unreasonable assumption related to stress testing is that market shocks occur instantly and that traders cannot re-hedge or adjust their positions.

When VaR is computed and analyzed, it is generally under more normalized market conditions, so it may not be accurate in a more stressful environment. A **stressed VaR** approach, which attempts to account for a significantly financial stressed period, has not been thoroughly tested or analyzed. Thus, VaR could lead to inaccurate risk assessment under market stresses.

Professor's Note: In Book 3, we will explain the calculation of stressed VaR.

Integrated Risk Measurement

LO 5.4: Compare unified and compartmentalized risk measurement.

Unified and compartmentalized risk measurement methods aggregate risks for banks. A compartmentalized approach sums risks separately, whereas a unified, or integrated, approach considers the interaction among risks.

A **unified approach** considers all risk categories simultaneously. This approach can capture possible compounding effects that are not considered when looking at individual risk measures in isolation. For example, unified approaches may consider market, credit, and operational risks all together.

When calculating capital requirements, banks use a **compartmentalized approach**, whereby capital requirements are calculated for individual risk types, such as market risk and credit risk. These stand-alone capital requirements are then summed in order to obtain the bank's overall level of capital.

The Basel regulatory framework uses a "building block" approach, whereby a bank's regulatory capital requirement is the sum of the capital requirements for various risk categories. Pillar 1 risk categories include market, credit, and operational risks. Pillar 2 risk categories incorporate concentration risks, stress tests, and other risks, such as liquidity, residual, and business risks.

Thus, the overall Basel approach to calculating capital requirements is a non-integrated approach to risk measurement. In contrast, an integrated approach would look at capital requirements for each of the risks simultaneously and account for potential risk correlations and interactions. Note that simply calculating individual risks and adding them together will not necessarily produce an accurate measure of true risk.

Risk Aggregation

LO 5.5: Compare the results of research on "top-down" and "bottom-up" risk aggregation methods.

A bank's assets can be viewed as a series of subportfolios consisting of market, credit, and operational risk. However, these risk categories are intertwined and at times difficult to separate. For example, foreign currency loans will contain both foreign exchange risk and credit risk. Thus, interactions among various risk factors should be considered.

The **top-down approach** to risk aggregation assumes that a bank's portfolio can be cleanly subdivided according to market, credit, and operational risk measures. In contrast, a **bottom-up approach** attempts to account for interactions among various risk factors.

In order to assess which approach is more appropriate, academic studies calculate the ratio of unified capital to compartmentalized capital (i.e., the ratio of integrated risks to separate risks). Top-down studies calculate this ratio to be less than one, which suggest that **risk diversification** is present and ignored by the separate approach. Bottom-up studies also often calculate this ratio to be less than one, however, this research has not been conclusive, and has recently found evidence of risk compounding, which produces a ratio greater than one. Thus, bottom-up studies suggest that risk diversification should be questioned.

It is conservative to evaluate market risk and credit risk independently. However, most academic studies confirm that market risk and credit risk should be looked at jointly. If a bank ignores risk interdependencies, a bank's capital requirement will be measured improperly due to the presence of risk diversification. Therefore, separate measurement of

market risk and credit risk most likely provides an upper bound on the integrated capital level.

Note that if a bank is unable to completely separate risks, the compartmentalized approach will not be conservative enough. Thus, the lack of complete separation could lead to an underestimation of risk. In this case, bank managers and regulators should conclude that the bank's overall capital level should be higher than the sum of the capital calculations derived from risks individually.

Balance Sheet Management

LO 5.6: Describe the relationship between leverage, market value of asset, and VaR within an active balance sheet management framework.

When a balance sheet is actively managed, the amount of leverage on the balance sheet becomes procyclical. This results because changes in market prices and risks force changes to risk models and capital requirements, which require adjustments to the balance sheet (i.e., greater risks require greater levels of capital). Thus, capital requirements tend to amplify boom and bust cycles (i.e., magnify financial and economic fluctuations). Academic studies have shown that balance sheet adjustments made through active risk management affect risk premiums and total financial market volatility.

Leverage (measured as total assets to equity) is inversely related to the market value of total assets. When net worth rises, leverage decreases, and when net worth declines, leverage increases. This results in a **cyclical feedback loop**: asset purchases increase when asset prices are rising, and assets are sold when asset prices are declining.

Value at risk is tied to a bank's level of economic capital. Given a target ratio of VaR to economic capital, a VaR constraint on leveraged investors can be established. An economic boom will relax this VaR constraint since a bank's level of equity is expanding. Thus, this expansion allows financial institutions to take on more risk and further increase debt. In contrast, an economic bust will tighten the VaR constraint and force investors to reduce leverage by selling assets when market prices and liquidity are declining. Therefore, despite increasingly sophisticated VaR models, current regulations intended to limit risk-taking have the potential to actually increase risk in financial markets.

Key Concepts

LO 5.1

The proper time horizon over which VaR is estimated depends on portfolio liquidity and the purpose for risk measurement. It is important to incorporate time-varying volatility into VaR models, because ignoring this factor could lead to an underestimation of risk. Backtesting VaR models is less effective over longer time horizons due to portfolio instability.

LO 5.2

Exogenous liquidity represents market-specific, average transaction costs. Endogenous liquidity is the adjustment for the price effect of liquidating specific positions. Endogenous liquidity risk is especially relevant in high-stress market conditions.

LO 5.3

VaR estimates the maximum loss that can occur given a specified level of confidence. It is a quantitative risk measure used by investment managers as a method to measure portfolio market risk. A downside of VaR is that it is not subadditive.

An alternative risk measure is expected shortfall, which is complex and computationally difficult. Spectral risk measures consider the investment manager's aversion to risk. These measures have select advantages over expected shortfall.

LO 5.4

Within a bank's risk assessment framework, a compartmentalized approach sums measured risks separately. A unified approach considers the interaction among various risk factors. Simply calculating individual risks and adding them together is not necessarily an accurate measure of true risk due to risk diversification. The Basel approach is a non-integrated approach to risk measurement.

LO 5.5

A top-down approach to risk assessment assumes that a bank's portfolio can be cleanly subdivided according to market, credit, and operational risk measures. To better account for the interaction among risk factors, a bottom-up approach should be used.

LO 5.6

When a balance sheet is actively managed, the amount of leverage on the balance sheet becomes procyclical. Leverage is inversely related to the market value of total assets. This results in a cyclical feedback loop. Financial institution capital requirements tend to amplify boom and bust cycles.

Concept Checkers

1. Which of the following statements is considered to be a drawback of the current Basel framework for risk measurement?
   A. Risk measurement focuses exclusively on VaR analysis.
   B. The current regulatory system encourages more risk-taking when times are good.
   C. There is not enough focus on a compartmentalized approach to risk assessment.
   D. There is not a feedback loop via the pricing of risk.

2. What type of liquidity risk is most troublesome for complex trading positions?
   A. Endogenous.
   B. Market-specific.
   C. Exogenous.
   D. Spectral.

3. Within the framework of risk analysis, which of the following choices would be considered most critical when looking at risks within financial institutions?
   A. Computing separate capital requirements for a bank's trading and banking books.
   B. Proper analysis of stressed VaR.
   C. Persistent use of backtesting.
   D. Consideration of interactions among risk factors.

4. What is a key weakness of the value at risk (VaR) measure? VaR:
   A. does not consider the severity of losses in the tail of the returns distribution.
   B. is quite difficult to compute.
   C. is subadditive.
   D. has an insufficient amount of backtesting data.

5. Which of the following statements is not an advantage of spectral risk measures over expected shortfall? Spectral risk measures:
   A. consider a manager's aversion to risk.
   B. are a special case of expected shortfall measures.
   C. have the ability to modify the risk measure to reflect an investor's specific risk aversion.
   D. have better smoothness properties when weighting observations.

Concept Checker Answers

1. **B** Institutions have a tendency to buy more risky assets when prices of assets are rising.

2. **A** Endogenous liquidity risk is especially relevant for complex trading positions.

3. **D** A unified approach is not used within the Basel framework, so the interaction among various risk factors is not considered when computing capital requirements for market, credit and operational risk; however, these interactions should be considered due to risk diversification.

4. **A** VaR does not consider losses beyond the VaR threshold level.

5. **B** Spectral risk measures consider aversion to risk and offer better smoothness properties. Expected shortfall is a special case of spectral risk measures.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Some Correlation Basics: Properties, Motivation, Terminology

Topic 6

Exam Focus

This topic focuses on the role correlation plays as an input for quantifying risk in multiple areas of finance. We will explain how correlation changes the value and risk of structured products such as credit default swaps (CDSs), collateralized debt obligations (CDOs), multi-asset correlation options, and correlation swaps. For the exam, understand how correlation risk is related to market risk, systemic risk, credit risk, and concentration ratios, and be familiar with how changes in correlation impact implied volatility, the value of structured products, and default probabilities. Also, be prepared to discuss how the misunderstanding of correlation contributed to the financial crisis of 2007 to 2009.

Financial Correlation Risk

LO 6.1: Describe financial correlation risk and the areas in which it appears in finance.

Correlation risk measures the risk of financial loss resulting from adverse changes in correlations between financial or nonfinancial assets. An example of financial correlation risk is the negative correlation between interest rates and commodity prices. If interest rates rise, losses occur in commodity investments. Another example of this risk occurred during the 2012 Greek crisis. The positive correlation between Mexican bonds and Greek bonds caused losses for investors of Mexican bonds.

The financial crisis beginning in 2007 illustrated how financial correlation risk can impact global markets. During this time period, correlations across global markets became highly correlated. Assets that previously had very low or negative correlations suddenly become very highly positively correlated and fell in value together.

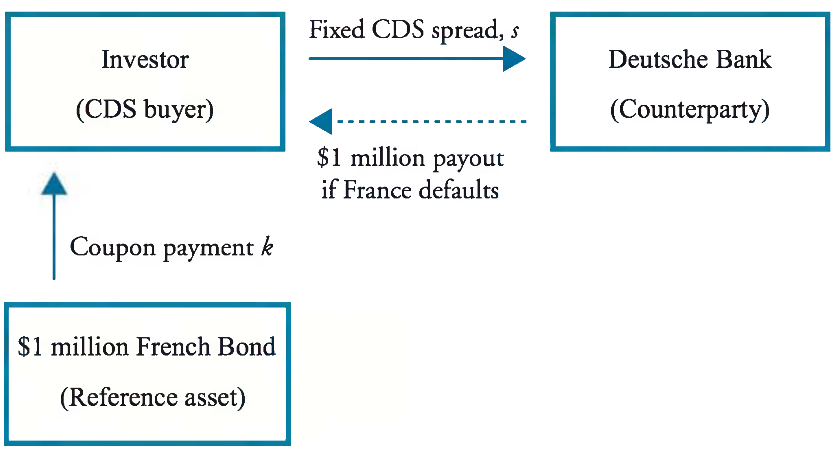
Nonfinancial assets can also be impacted by correlation risk. For example, the correlation of sovereign debt levels and currency values can result in financial losses for exporters. In 2012, U.S. exporters experienced losses due to the devaluation of the euro. Similarly, a low gross domestic product (GDP) for the United States has major adverse impacts on Asian and European exporters who rely heavily on the U.S. market. Another nonfinancial example is related to political events, such as uprisings in the Middle East that cause airline travel to decrease due to rising oil prices.

Financial correlations can be categorized as static or dynamic. **Static financial correlations** do not change and measure the relationship between assets for a specific time period. Examples of static correlation measures are value at risk (VaR), correlation copulas for collateralized debt obligations (CDOs), and the binomial default correlation model. **Dynamic financial correlations** measure the comovement of assets over time. Examples of dynamic financial correlations are pairs trading, deterministic correlation approaches, and stochastic correlation processes.

Structured products are becoming an increasing area of concern regarding correlation risk. The following example demonstrates the role correlation risk plays in **credit default swaps** (CDS). A CDS transfers credit risk from the investor (CDS buyer) to a counterparty (CDS seller).

Suppose an investor purchases $1 million of French bonds and is concerned about France defaulting. The investor (CDS buyer) can transfer the default risk to a counterparty (CDS seller). Figure 1 illustrates the process for an investor transferring credit default risk by purchasing a CDS from Deutsche Bank (a large European bank).

Figure 1: CDS Buyer Hedging Risk in Foreign Bonds



Assume the recovery rate is zero with no accrued interest in the event of default. The investor (CDS buyer) is protected if France defaults because the investor receives a $1 million payment from Deutsche Bank. The fixed CDS spread is valued based on the default probability of the reference asset (French Bond) and the joint default correlation of Deutsche Bank and France. A paper loss occurs if the correlation risk between Deutsche Bank and France increases because the value of the CDS will decrease. If Deutsche Bank and France default (worst case scenario), the investor loses the entire $1 million investment.

If there is positive correlation risk between Deutsche Bank and France, the investor has **wrong-way risk** (WWR). The higher the correlation risk, the lower the CDS spread, s. The increasing correlation risk increases the probability that both the French bond (reference asset) and Deutsche Bank (counterparty) default.

The dependencies between the CDS spread, s, and correlation risk may be *nonmonotonous*. This means that the CDS spread may sometimes increase and sometimes decrease if correlation risk increases. For example, for a correlation of –1 to –0.4, the CDS spread may

increase slightly. This is due to the fact that a high negative correlation implies either France or Deutsche Bank will default, but not both. If France defaults, the $1 million is recovered from Deutsche Bank. If Deutsche Bank defaults, the investor loses the value of the CDS spread and the investor will need to repurchase a CDS spread to hedge the position. The new CDS spread cost will most likely increase in the event that Deutsche Bank defaults or if the credit quality of France decreases.

There are many areas in finance that have financial correlations. Five common finance areas where correlations play an important role are (1) investments, (2) trading, (3) risk management, (4) global markets, and (5) regulation.

Correlations in Financial Investments

In 1952, Harry Markowitz provided the foundation of modern investment theory by demonstrating the role that correlation plays in reducing risk. The portfolio return is simply the weighted average of the individual returns where the weights are the percentage of investment in each asset. The following equation defines the average return (i.e., **mean**) for a portfolio, μ_P, comprised of assets *X* and *Y*. Asset *X* has a weight of w_X and an average return of μ_X, and asset *Y* has a weight of w_Y and an average return of μ_Y.

$$\mu_P = w_X\mu_X + w_Y\mu_Y$$

The **standard deviation** of a portfolio is determined by the variances of each asset, the weights of each asset, and the covariance between assets. The risk or standard deviation (i.e., volatility) for a two-asset portfolio is calculated as follows:

$$\sigma_P = \sqrt{w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + 2w_Xw_Y\,\text{cov}_{XY}}$$

Let us review how variances, covariance, and correlation are calculated using the following example. Suppose an analyst gathers historical prices for two assets, *X* and *Y*, and calculates their average returns as illustrated in Figure 2.

Figure 2: Prices and Returns for Assets *X* and *Y*

Year	*X*	*Y*	*Return X*	*Return Y*
2009	90	150		
2010	120	180	0.3333	0.2000
2011	105	340	(0.1250)	0.8889
2012	170	320	0.6190	(0.0588)
2013	150	360	(0.1176)	0.1250
2014	270	310	0.8000	(0.1389)
		Average Return	0.3019	0.2032

The calculations for determining the standard deviations, variances, covariance, and correlation for assets *X* and *Y* are illustrated in Figure 3.

Figure 3: Variances and Covariance for Assets *X* and *Y*

Year	*Return X*	*Return Y*	$X_t - \mu_X$	$Y_t - \mu_Y$	$(X_t - \mu_X)^2$	$(Y_t - \mu_Y)^2$	$(X_t - \mu_X) \times (Y_t - \mu_Y)$
2010	0.3333	0.2000	0.0314	(0.0032)	0.0010	0.0000	(0.0001)
2011	(0.1250)	0.8889	(0.4269)	0.6857	0.1823	0.4701	(0.2927)
2012	0.6190	(0.0588)	0.3171	(0.2621)	0.1006	0.0687	(0.0831)
2013	(0.1176)	0.1250	(0.4196)	(0.0782)	0.1761	0.0061	0.0328
2014	0.8000	(0.1389)	0.4981	(0.3421)	0.2481	0.1170	(0.1704)
Mean	**0.3019**	**0.2032**			0.7079	0.6620	(0.5135)
				Variance	**0.1770**	**0.1655**	**(0.1284)**
				Standard Deviation	**0.4207**	**0.4068**	
				Correlation	**(0.7501)**		

Notice that the sixth and seventh columns of Figure 3 are used to calculate the variance of *X* and *Y*, respectively. The deviation from each respective mean is squared to calculate the variance for each asset: $(X_t - \mu_X)^2$ for *X* and $(Y_t - \mu_Y)^2$ for *Y*. The sum of the deviations is then divided by four (i.e., the number of observations minus one for degrees of freedom). For example, the asset *X* variance is calculated by taking 0.7079 and dividing by 4 (i.e., n – 1) to get 0.1770.

Covariance is a measure of how two assets move together over time. The last column of Figure 3 illustrates that the calculation for covariance is similar to the calculation for variance. However, instead of squaring each deviation from the mean, the last column multiplies the deviations from the mean for each respective asset together. This not only captures the magnitude of movement but also the direction of movement. Thus, when asset returns are moving in opposite directions for the same time period, the product of their deviations is negative. The following equation defines the calculation for covariance. The sum of the products of the deviations from the means is –0.5135 in the last column of Figure 3. Covariance is calculated as –0.1284 by dividing –0.5135 by 4 (i.e., n – 1).

$$\text{cov}_{XY} = \frac{\sum_{t=1}^{n}(X_t - \mu_X)(Y_t - \mu_Y)}{n-1}$$

In finance, the **correlation coefficient** is often used to standardize the comovement or covariance between assets. The following equation defines the correlation for two assets, *X* and *Y*, by dividing covariance, cov_{XY}, by the product of the asset standard deviations, $\sigma_X\sigma_Y$.

$$\rho_{XY} = \frac{\text{cov}_{XY}}{\sigma_X\sigma_Y}$$

The correlation in this example is –0.7501, which is calculated as:

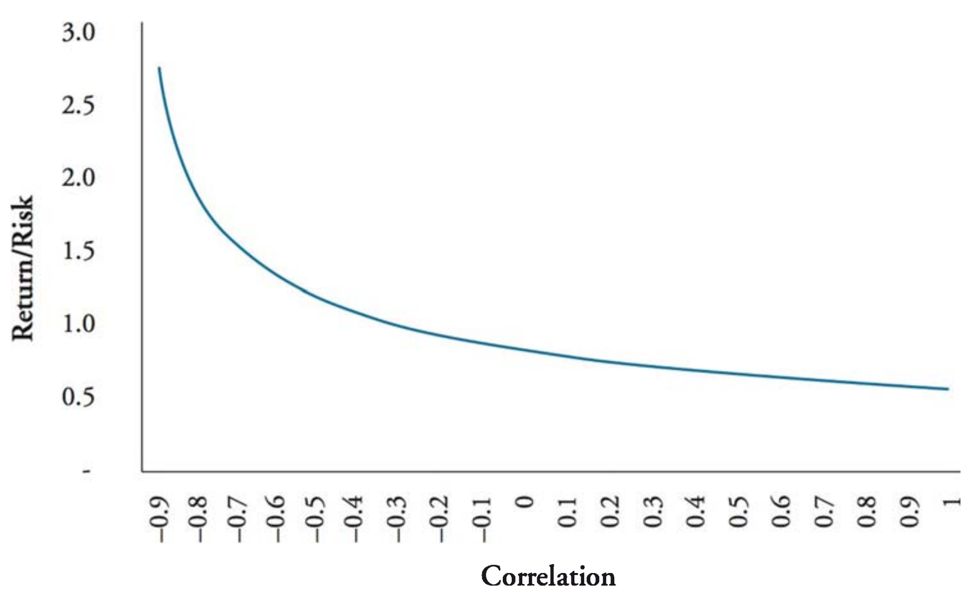
$$-0.1284 / (0.4207 \times 0.4068) = -0.7501$$

In his research, Markowitz emphasized the importance of focusing on risk-adjusted returns. The return/risk ratio measures the average return for a portfolio, μ_P, by the risk of the portfolio, σ_P. Figure 2 provided the average return for X and Y as 0.3019 and 0.2032, respectively. If we assume the portfolio is equally weighted, the average return for the portfolio is 0.2526, the correlation between assets X and Y is –0.7501, and the standard deviations for X and Y are 0.4207 and 0.4068, respectively. The standard deviation for an equally-weighted portfolio is determined using the following expression:

$$\sqrt{\left(0.5^2 \times 0.4207^2\right)+\left(0.5^2 \times 0.4068^2\right)+(2 \times 0.5 \times 0.5 \times -0.1284)}$$
$$=\sqrt{0.02142}=0.1464$$

The return/risk ratio of this equally-weighted two-asset portfolio is 1.725 (calculated as 0.2526 divided by 0.1464). Figure 4 illustrates the relationship of the return/risk ratio and correlation. The lower the correlation between the two assets, the higher the return/risk ratio. A very high negative correlation (e.g., –0.9) results in a return/risk ratio greater than 250%. A very high positive correlation (e.g., +0.9) results in a return/risk ratio near 50%.

Figure 4: Relationship of Return/Risk Ratio and Correlation



Correlation in Trading with Multi-Asset Options

Correlation trading strategies involve trading assets that have prices determined by the comovement of one or more assets over time. **Correlation options** have prices that are very sensitive to the correlation between two assets and are often referred to as *multi-asset options*.

A quick review of the common notation for options is helpful. Assume the price of asset one and two are noted as S_1 and S_2, respectively, and that the strike price, K, for a call option is the predetermined price an asset can be purchased. Likewise, the strike price, K, for a put option is the predetermined price an asset can be sold for.

The correlation between the two assets S_1 and S_2 is an important factor in determining the price of correlation options. Figure 5 lists a number of multi-asset correlation strategies along with their payoffs. For all of these strategies, a lower correlation results in a higher option price. A low correlation is expected to result in one asset price going higher while the other is lower. Thus, there is a better chance of a higher payout.

Figure 5: Payoffs for Multi-Asset Correlation Strategies

Correlation strategies	Payoff
Option on higher of two stocks	$\max(S_1, S_2)$
Call option on maximum of two stocks	$\max[0, \max(S_1, S_2) - K]$
Exchange option	$\max(0, S_2 - S_1)$
Spread call option	$\max(0, S_2 - S_1 - K)$
Dual-strike call option	$\max(0, S_1 - K_1, S_2 - K_2)$
Portfolio of basket options	$\max\left[\sum_{i=1}^{n} n_i \times S_i - K, 0\right]$, where n_i = weight of asset i

Another correlation strategy that is not listed in Figure 5 is a correlation option on the worse of two stocks where the payoff is the minimum of the two stock prices. This is the only correlation option where a lower correlation is not desirable because it reduces the correlation option price.

We can better understand the role correlation plays by taking a closer look at the valuation of the exchange option. The exchange option has a payoff of $\max(0, S_2 - S_1)$. The buyer of the option has the right to receive asset 2 and give away asset 1 when the option matures. The standard deviation of the exchange option, σ_E, is the implied volatility of S_2 / S_1, which is defined as:

$$\sigma_E = \sqrt{\sigma_X^2 + \sigma_Y^2 - 2\text{cov}_{XY}}$$

Implied volatility is an important determinant of the option's price. Thus, the exchange option price is highly sensitive to the covariance or correlation between the two assets. The price of the exchange option is close to zero when the correlation is close to 1 because the two asset prices move together, and the spread between them does not change. The price of the exchange option increases as the correlation between the two assets decreases because the spread between the two assets is more likely to be greater.

Quanto Option

The quanto option is another investment strategy using correlation options. It protects a domestic investor from foreign currency risk. However, the financial institution selling the

quanto call does not know how deep in the money the call will be or what the exchange rate will be when the option is exercised to convert foreign currency to domestic currency. Lower correlations between currencies result in higher prices for quanto options.

Example: Quanto option

Suppose a U.S. investor buys a quanto call to invest in the Nikkei index and protect potential gains by setting a fixed currency exchange rate (USD/JPY). How does the correlation between the call on the Nikkei index and the exchange rate impact the price of the quanto option?

Answer:

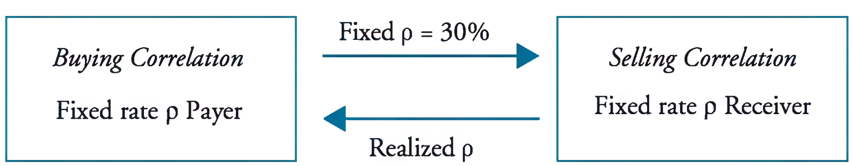
The U.S. investor buys a quanto call on the Nikkei index that has a fixed exchange rate for converting yen to dollars. If the correlation coefficient is positive (negative) between the Nikkei index and the yen relative to the dollar, an increasing Nikkei index results in an increasing (decreasing) value of the yen. Thus, the lower the correlation, the higher the price for the quanto option. If the Nikkei index increases and the yen decreases, the financial institution will need more yen to convert the profits in yen from the Nikkei investment into dollars.

Correlation Swap

LO 6.3: Describe the structure, uses, and payoffs of a correlation swap.

A correlation swap is used to trade a fixed correlation between two or more assets with the correlation that actually occurs. The correlation that will actually occur is unknown and is referred to as the *realized* or *stochastic correlation*. Figure 6 illustrates how a correlation swap is structured. In this example, the party buying a correlation swap pays a fixed correlation rate of 30%, and the entity selling a correlation receives the fixed correlation of 30%.

Figure 6: Correlation Swap with a Fixed Correlation Rate



The present value of the correlation swap increases for the correlation buyer if the realized correlation increases. The following equation calculates the realized correlation that actually occurs over the time period of the swap for a portfolio of *n* assets, where $\rho_{i,j}$ is the correlation coefficient:

$$\rho_{\text{realized}} = \frac{2}{n^2 - n} \sum_{i>j} \rho_{i,j}$$

The payoff for the investor buying the correlation swap is calculated as follows:

notional amount × ($\rho_{realized} - \rho_{fixed}$)

Example: Correlation swap

Suppose a correlation swap buyer pays a fixed correlation rate of 0.2 with a notional value of $1 million for one year for a portfolio of three assets. The realized pairwise correlations of the daily log returns [$\ln(S_t / S_{t-1})$] at maturity for the three assets are $\rho_{2,1} = 0.6$, $\rho_{3,1} = 0.2$, and $\rho_{3,2} = 0.04$. (Note that for all pairs i > j.) What is the correlation swap buyer's payoff?

Answer:

The realized correlation is calculated as:

$$\rho_{realized} = \frac{2}{3^2 - 3} \times (0.6 + 0.2 + 0.04) = 0.28$$

The payoff for the correlation swap buyer is then calculated as:

$1,000,000 × (0.28 – 0.20) = $80,000

Another example of buying correlation is to buy call options on a stock index (such as the Standard & Poor's 500 Index) and sell call options on individual stocks held within the index. If correlation increases between stocks within the index, this causes the implied volatility of call options to increase. The increase in price for the index call options is expected to be greater than the increase in price for individual stocks that have a short call position.

An investor can also buy correlation by paying fixed in a variance swap on an index and receiving fixed on individual securities within the index. An increase in correlation for securities within the index causes the variance to increase. An increase in variance causes the present value of the position to increase for the fixed variance swap payer (i.e., variance swap buyer).

Risk Management

LO 6.4: Estimate the impact of different correlations between assets in the trading book on the VaR capital charge.

The primary goal of risk management is to mitigate financial risk in the form of market risk, credit risk, and operational risk. A common risk management tool used to measure market risk is **value at risk** (VaR). VaR for a portfolio measures the potential loss in value

for a specific time period for a given confidence level. The formula for calculating VaR using the **variance-covariance method** (a.k.a. delta-normal method) is shown as follows:

$$VaR_P = \sigma_P \alpha \sqrt{x}$$

In this equation, σ_P is the daily volatility of the portfolio, α is the *z*-value from the standard normal distribution for a specific confidence level, and *x* is the number of trading days. The volatility of the portfolio, σ_P, includes a measurement of correlation for assets within the portfolio defined as:

$$\sigma_P = \sqrt{\beta_h \times C \times \beta_v}$$

where:
β_h = horizontal β vector of investment amount
C = covariance matrix of returns
β_v = vertical β vector of investment amount

Example: Computing VaR with the variance-covariance method

Assume you have a two-asset portfolio with $8 million in asset A and $4 million in asset B. The portfolio correlation is 0.6, and the daily standard deviation of returns for assets A and B are 1.5% and 2%, respectively. What is the 10-day VaR of this portfolio at a 99% confidence level (i.e., α = 2.33)?

Answer:

The first step in solving for the 10-day VaR requires constructing the covariance matrix.

$$\begin{aligned}
cov_{11} &= \sigma_1^2 = 0.015^2 = 0.000225 \\
cov_{22} &= \sigma_2^2 = 0.02^2 = 0.0004 \\
cov_{12} &= \rho_{12} \times \sigma_1 \times \sigma_2 = 0.6 \times 0.015 \times 0.02 = 0.00018
\end{aligned}$$

Thus, the covariance matrix, C, can be represented as:

$$\begin{pmatrix} cov_{11} & cov_{12} \\ cov_{21} & cov_{22} \end{pmatrix} = \begin{pmatrix} 0.000225 & 0.00018 \\ 0.00018 & 0.0004 \end{pmatrix}$$

Next, the standard deviation of the portfolio, σ_P, is determined by first solving for $\beta_h \times C$, then solving for $(\beta_h \times C) \times \beta_v$, and finally taking the square root of the second step.

Step 1: Compute $\beta_h \times C$:

$$\begin{aligned}
&\begin{bmatrix} 8 & 4 \end{bmatrix} \begin{pmatrix} 0.000225 & 0.00018 \\ 0.00018 & 0.0004 \end{pmatrix} \\
&= \begin{bmatrix} (8 \times 0.000225) + (4 \times 0.00018) & (8 \times 0.00018) + (4 \times 0.0004) \end{bmatrix} \\
&= \begin{bmatrix} 0.00252 & 0.00304 \end{bmatrix}
\end{aligned}$$

Step 2: Compute $(\beta_h \times C) \times \beta_v$:

$$\begin{bmatrix}0.00252 & 0.00304\end{bmatrix}\begin{bmatrix}8\\4\end{bmatrix}$$
$$= (0.00252 \times 8) + (0.00304 \times 4) = 0.03232$$

Step 3: Compute σ_P:

$$\sigma_P = \sqrt{\beta_h \times C \times \beta_v} = \sqrt{0.03232} = 0.1798 \text{ or } 17.98\%$$

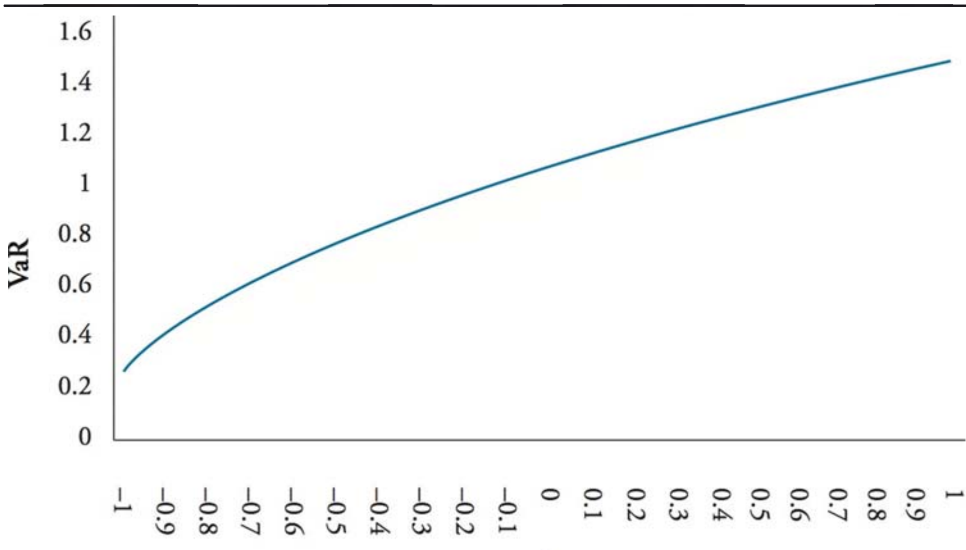
The 10-day portfolio VaR (in millions) at the 99% confidence level is then computed as:

$$\text{VaR}_P = \sigma_P \alpha \sqrt{x} = 0.1798 \times 2.33 \times \sqrt{10} = 1.3248$$

This suggests that the loss will only exceed $1,324,800 once every 100 10-day periods. This is approximately once every 1,000 trading days or once every four years assuming there are 250 trading days in a year.

Figure 7 illustrates the relationship between correlation and VaR for the previous two-asset portfolio example. The VaR for the portfolio increases as the correlation between the two assets increases.

Figure 7: Relationship Between VaR and Correlation for Two-Asset Portfolio



The Basel Committee on Banking Supervision (BCBS) requires banks to hold capital based on the VaR for their portfolios. The BCBS requires banks to hold capital for assets in the trading book of at least three times greater than 10-day VaR. The trading book includes assets that are marked-to-market, such as stocks, futures, options, and swaps. The bank in the previous example would be required by the Basel Committee to hold capital of:

$1,324,800 × 3 = $3,974,400

Correlations During the Recent Financial Crisis

LO 6.2: Explain how correlation contributed to the global financial crisis of 2007 to 2009.

The correlations of assets within and across different sectors and geographical regions were a major contributing factor for the financial crisis of 2007 to 2009. The economic environment, risk attitude, new derivative products, and new copula correlation models all contributed to the crisis.

Investors became more risk averse shortly after the internet bubble that began in the 1990s. The economy and risk environment was recovering with low credit spreads, low interest rates, and low volatility. The overly optimistic housing market led individuals to take on more debt on overvalued properties. New structured products known as collateralized debt obligations (CDOs), constant-proportion debt obligations (CPDOs), and credit default swaps (CDSs) helped encourage more speculation in real estate investments. Rating agencies, risk managers, and regulators overlooked the amount of leverage individuals and financial institutions were taking on. All of these contributing factors helped set the stage for the financial crisis that would be set off initially by defaults in the subprime mortgage market.

Risk managers, financial institutions, and investors did not understand how to properly measure correlation. Risk managers used the newly developed copula correlation model for measuring correlation in structured products. It is common for CDOs to contain up to 125 assets. The copula correlation model was designed to measure $[n \times (n - 1) / 2]$ assets in structured products. Thus, risk managers of CDOs needed to estimate and manage 7,750 correlations (i.e., $125 \times 124 / 2$).

CDOs are separated into several tranches based on the degree of default risk. The riskiest tranche is called the equity tranche, and investors in this tranche are typically exposed to the first 3% of defaults. The next tranche is referred to as the mezzanine tranche where investors are typically exposed to the next 4% of defaults (above 3% to 7%). The copula correlation model was trusted to monitor the default correlations across different tranches. A number of large hedge funds were short the CDO equity tranche and long the CDO mezzanine tranche. In other words, potential losses from the equity tranche were thought to be hedged with gains from the mezzanine tranche. Unfortunately, huge losses lead to bankruptcy filings by several large hedge funds because the correlation properties across tranches were not correctly understood.

Correlation played a key role in the bond market for U.S. automobile makers and the CDO market just prior to the financial crisis. A junk bond rating typically leads to major price decreases as pension funds, insurance companies, and other large financial institutions sell their holdings and are not willing to hold non-investment grade bonds. Bonds within specific credit quality levels typically are more highly correlated. Bonds across credit quality levels typically have lower correlations.

Rating agencies downgraded General Motors and Ford to junk bond status in May of 2005. Following the change in bond ratings for Ford and General Motors, the equity tranche spread increased dramatically. This caused losses for hedge funds that were short the equity tranche (i.e., sold credit protection). At the same time, the correlations decreased for CDOs of investment grade bonds. The lower correlations in the mezzanine tranche led to losses for hedge funds that were long the mezzanine tranche.

The CDO market, comprised primarily of residential mortgages, increased from $64 billion in 2003 to $455 billion in 2006. Liberal lending policies combined with overvalued real estate created the perfect storm in the subprime mortgage market. Housing prices became stagnate in 2006 leading to the first string of mortgage defaults. In 2007, the real estate market collapsed as the number of mortgage defaults increased. The CDO market, which was linked closely to mortgages, collapsed as well. This led to a global crisis as stock and commodities markets collapsed around the world. As a result, correlations in stock markets increased as the U.S. stock market crashed. Default correlations in CDO markets and bond markets also increased as the value of real estate and financial stability of individuals and institutions was highly questionable.

The CDO equity tranche spread typically decreases when default correlations increase. A lower equity tranche spread typically leads to an increase in value of the equity tranche. Unfortunately, the probability of default in the subprime market increased so dramatically in 2007 that it lowered the value of all CDO tranches. Thus, the default correlations across CDO tranches increased. The default rates also increased dramatically for all residential mortgages. Even the highest quality CDO tranches with AAA ratings lost 20% of their value as they were no longer protected from the lower tranches. The losses were even greater for many institutions with excess leverage in the senior tranches that were thought to be safe havens. The leverage in the CDO market caused risk exposures for investors to be 10 to 20 times higher than the investments.

In addition to the rapid growth in the CDO market, the credit default swap (CDS) market grew from $8 trillion to $60 trillion during the 2004 to 2007 time period. As mentioned earlier, CDSs are used to hedge default risk. CDSs are similar to insurance products as the risk exposure in the debt market is transferred to a broader market. The CDS seller must be financially stable enough to protect against losses. The recent financial crisis revealed that American International Group (AIG) was overextended, selling $500 billion in CDSs with little reinsurance. Also, Lehman Brothers had leverage 30.7 times greater than equity in September 2008 leading to its bankruptcy. However, the leverage was much higher considering the large number of derivatives transactions that were also held with 8,000 different counterparties.

Regulators are in the process of developing Basel III in response to the financial crisis. New standards for liquidity and leverage ratios for financial institutions are also being implemented. New correlation models are being developed and implemented such as the Gaussian copula, credit value adjustment (CVA) for correlations in derivatives transactions, and wrong-way risk (WWR) correlation. These new models hope to address correlated defaults in multi-asset portfolios.

The Role of Correlation Risk in Other Types of Risk

LO 6.5: Explain the role of correlation risk in market risk and credit risk.

LO 6.6: Relate correlation risk to systemic and concentration risk.

A major concern for risk managers is the relationship between correlation risk and other types of risk such as market, credit, systemic, and concentration risk. Examples of major factors contributing to market risk are interest rate risk, currency risk, equity price risk, and commodity risk. As discussed earlier, risk managers typically measure *market risk* in terms of VaR. Because the covariance matrix of assets is an important input of VaR, correlation risk is extremely important. Another important risk management tool used to quantify market risk is **expected shortfall** (ES). Expected shortfall measures the impact of market risk for extreme events or tail risk. Given that correlation risk refers to the risk that the correlation between assets changes over time, the concern is how the covariance matrix used for calculating VaR or ES changes over time due to changes in market risk.

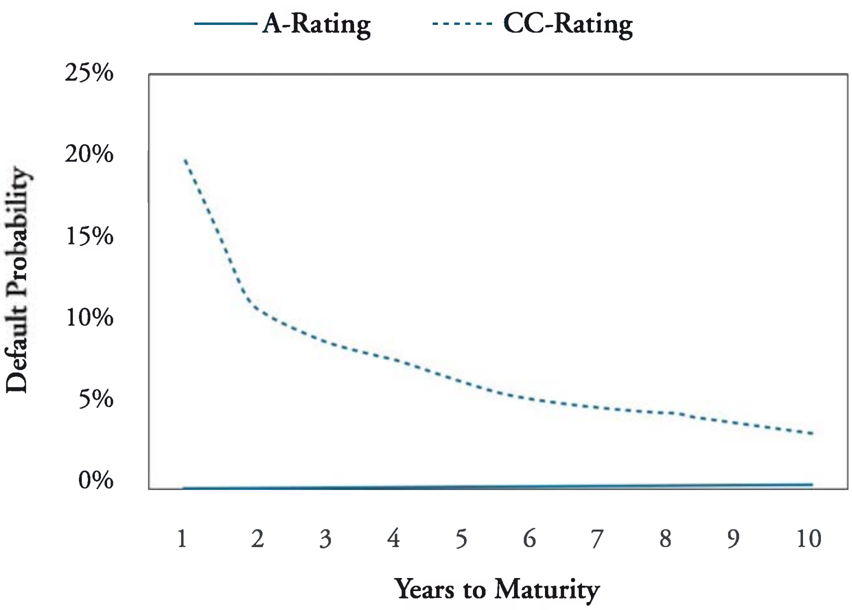
Risk managers are also concerned with measuring *credit risk* with respect to migration risk and default risk. **Migration risk** is the risk that the quality of a debtor decreases following the lowering of quality ratings. Lower debt quality ratings imply higher default probabilities. When a debt rating decreases, the present value of the underlying asset decreases, which creates a paper loss. As discussed previously, correlation risk between a reference asset and counterparty (CDS seller) is an important concern for investors. A higher correlation increases the probability of total loss of an investment.

Financial institutions such as mortgage companies and banks provide a variety of loans to individuals and entities. **Default correlation** is of critical importance to financial institutions in quantifying the degree that defaults occur at the same time. A lower default correlation is associated with greater diversification of credit risk. Empirical studies have examined historical default correlations across and within industries. Most default correlations across industries are positive with the exception of the energy sector. The energy sector has little or no correlation with other sectors and is, therefore, more resistant to recessions.

Historical data suggests that default correlations are higher within industries. This finding implies that systematic factors impacting the overall market and credit risk have much more influence in defaults than individual or company-specific factors. For example, if Chrysler defaults, then Ford and General Motors are more likely to default and have losses rather than benefit from increased market share. Thus, commercial banks limit exposures within a specific industry. The key point is that creditors benefit by diversifying exposure across industries to lower the default correlations of debtors.

Risk managers can also use a term structure of defaults to analyze credit risk. Rating agencies such as Moody's provide default probabilities based on bond ratings and time to maturity as illustrated in Figure 8.

Figure 8: Default Term Structure for A- and CC-Rated Bonds



Notice in Figure 8 that the default term structure increases slightly with time to maturity for most investment grade bonds (solid line). This is expected because bonds are more likely to default as many market or company factors can change over a longer time period. Conversely, for non-investment grade bonds (dashed line), the probability of default is higher in the immediate time horizon. If the company survives the near-term distressed situation, the probability of default decreases over time.

Lehman Brothers filed for bankruptcy in September of 2008. This bankruptcy event was an important signal of the severity of the financial crisis and the level of systemic risk. **Systemic risk** refers to the potential risk of a collapse of the entire financial system. It is interesting to examine the extent of the stock market crash that began in October 2007. From October 2007 to March 2009, the Dow Jones Industrial Average fell over 50% and only 11 stocks increased in the entire Standard & Poor's 500 Index (S&P 500). The decrease in value of 489 stocks in the S&P 500 during this time period reflected how a systemic financial crisis impacts the economy with decreasing disposable income for individuals, decreasing GDP, and increasing unemployment.

The sectors represented in the 11 increasing stocks were consumer stables (Family Dollar, Ross Stores, and Walmart), educational (Apollo Group and DeVry Inc.), pharmaceuticals (Edward Lifesciences and Gilead Pharmaceuticals), agricultural (CF Industries), entertainment (Netflix), energy (Southwestern Energy), and automotive (AutoZone). The consumer staples and pharmaceutical sector are often recession resistant as individuals continue to need basic necessities such as food, household supplies, and medications. The educational sector is also resilient as more unemployed workers go back to school for education and career changes.

Studies examined the relationship between the correlations of stocks in the U.S. stock market and the overall market during the 2007 crisis. From August of 2008 to March of 2009, there was a freefall in the U.S. equity market. During this same time period, correlations of stocks with each other increased dramatically from a pre-crisis average

correlation level of 27% to over 50%. Thus, when diversification was needed most during the financial crisis, almost all stocks become more highly correlated and, therefore, less diversified. The severity of correlation risk is even greater during a systemic crisis when one considers the higher correlations of U.S. equities with bonds and international equities.

Concentration risk is the financial loss that arises from the exposure to multiple counterparties for a specific group. Concentration risk is measured by the **concentration ratio.** A lower (higher) concentration ratio reflects that the creditor has more (less) diversified default risk. For example, the concentration ratio for a creditor with 100 loans of equal size to different entities is 0.01 (= 1 / 100). If a creditor has one loan to one entity, the concentration ratio for the creditor is 1.0 (= 1 / 1). Loans can be further analyzed by grouping them into different sectors. If loan defaults are more highly correlated within sectors, when one loan defaults within a specific sector, it is more likely that another loan within the same sector will also default. The following examples illustrate the relationship between concentration risk and correlation risk.

Example: Concentration ratio for bank X and one loan to company A

Suppose commercial bank X makes a $5 million loan to company A, which has a 5% default probability. What is the concentration ratio and expected loss (EL) for commercial bank X under the worst case scenario? Assume loss given default (LGD) is 100%.

Answer:

Commercial bank X has a concentration ratio of 1.0 because there is only one loan. The worst case scenario is that company A defaults resulting in a total loss of loan value. Given that there is a 5% probability that company A defaults, EL for commercial bank X is $250,000 (= 0.05 × 5,000,000).

Example: Concentration ratio for bank Y and two loans to companies A and B

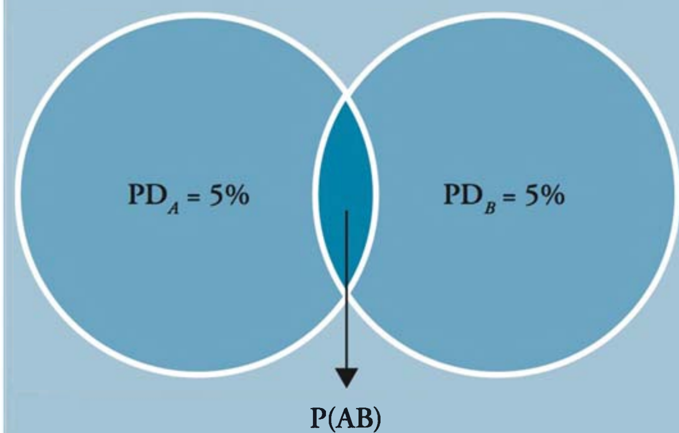
Suppose commercial bank Y makes a $2,500,000 loan to company A and a $2,500,000 loan to company B. Assuming companies A and B each have a 5% default probability, what is the concentration ratio and expected loss (EL) for commercial bank Y under the worst case scenario? Assume default correlation between companies is 1.0 and loss given default (LGD) is 100%.

Answer:

Commercial bank Y has a concentration ratio of 0.5 (calculated as 1 / 2). The expected loss for commercial bank Y depends on the default correlation of companies A and B. Note that changes in the concentration ratio are directly related to changes in the default correlations. A decrease in the concentration ratio results in a decrease in the default correlation. The default of companies A and B can be expressed as two binomial events with a value of 1 in default and 0 if not in default.

Figure 9 illustrates the joint probability that both companies A and B are in default, P(AB).

Figure 9: Joint Probability of Default for Companies A and B



The following equation computes the joint probability that both companies A and B are in default at the same time:

$$P(AB) = \rho_{AB}\sqrt{PD_A(1-PD_A)\times PD_B(1-PD_B)} + PD_A \times PD_B$$

where:

ρ_{AB} = default correlation coefficient for A and B

$\sqrt{PD_A(1-PD_A)}$ = standard deviation of the binomial event A

The default probability of company A is 5%. Thus, the standard deviation for company A is:

$$\sqrt{0.05(1-0.05)} = 0.2179$$

Company B also has a default probability of 5% and, therefore, will also have a standard deviation of 0.2179. We can now calculate the expected loss under the worst case scenario where both companies A and B are in default. Assuming that the default correlation between A and B is 1.0, the joint probability of default is:

$$P(AB) = 1.0\sqrt{0.05(0.95) \times 0.05(0.95)} + 0.05 \times 0.05$$
$$= 1.0\sqrt{0.00226} + 0.0025 = 0.05$$

If the default correlation between companies A and B is 1.0, the expected loss for commercial bank Y is $250,000 (0.05 × $5,000,000). Notice that when the default correlation is 1.0, this is the same as making a $5 million loan to one company.

Now, let's assume that the default correlation between companies A and B is 0.5. What is the expected loss for commercial bank Y? The joint probability of default for A and B, assuming a default correlation of 0.5, is:

$$P(AB) = 0.5\sqrt{0.00226} + 0.0025 = 0.02625$$

Thus, the expected loss for the worst case scenario for commercial bank Y is:

$$EL = 0.02625 \times \$5{,}000{,}000 = \$131{,}250$$

If we assume the default correlation coefficient is 0, the joint probability of default is 0.0025 and the expected loss for commercial bank Y is only $12,500. Thus, a lower default correlation results in a lower expected loss under the worst case scenario.

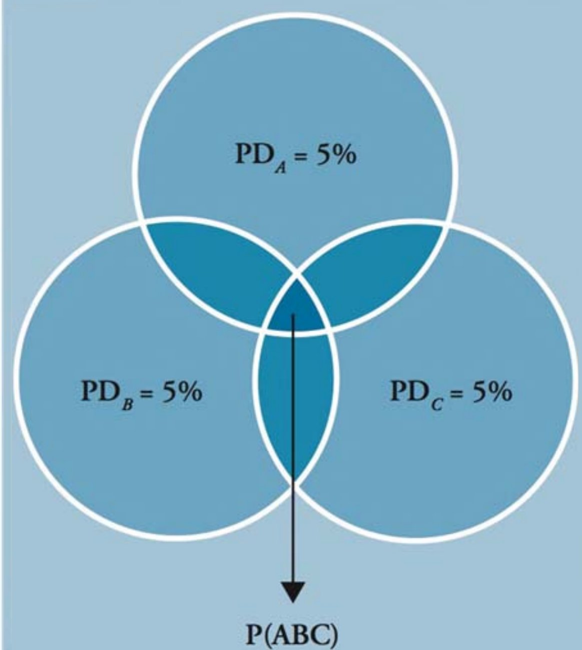
Example: Concentration ratio for bank Z and three loans to companies A, B, and C

Now we can examine what happens to the joint probability of default (i.e., the worst case scenario) if the concentration ratio is reduced further. Suppose that commercial bank Z makes three $1,666,667 loans to companies A, B, and C. Also assume the default probability for each company is 5%. What is the concentration ratio for commercial bank Z, and how will the joint probability be impacted?

Answer:

Commercial bank Z has a concentration ratio of 0.333 (calculated as 1 / 3). Figure 10 illustrates the joint probability of all three loans defaulting at the same time, P(ABC) (i.e., the small area in the center of Figure 10 where all three default probabilities overlap). Note that as the concentration ratio decreases, the joint probability also decreases.

Figure 10: Joint Probability of Default for Companies A, B, and C



Professor Note: The assigned reading did not cover the calculation of the joint probability for three binomial events occurring. The focus here is on understanding that as the concentration ratio decreases, the probability of the worst case scenario also decreases. Both a lower concentration ratio and lower correlation coefficient reduce the joint probability of default.

Key Concepts

LO 6.1

Correlation risk measures the risk of financial loss resulting from adverse changes in correlations between financial or nonfinancial assets. For example, financial correlation risk can result from the negative correlation between interest rates and commodity prices. For almost all correlation option strategies, a lower correlation results in a higher option price.

LO 6.2

In May of 2005, several large hedge funds had losses on both sides of a hedged position short the collateralized debt obligation (CDO) equity tranche spread and long the CDO mezzanine tranche. The decrease in default correlations in the mezzanine tranche led to losses in the mezzanine tranche.

American International Group (AIG) and Lehman Brothers were highly leveraged in credit default swaps (CDSs) during the recent financial crisis. Their financial troubles revealed the impact of increasing default correlations with tremendous leverage.

LO 6.3

A correlation swap is used to trade a fixed correlation between two assets with the realized correlation. The payoff for the investor buying the correlation swap is:

$$\text{notional amount} \times (\rho_{\text{realized}} - \rho_{\text{fixed}})$$

where:

$$\rho_{\text{realized}} = \frac{2}{n^2 - n} \sum_{i>j} \rho_{i,j}$$

LO 6.4

Value at risk (VaR) for a portfolio measures the potential loss in value for a specific time period for a given confidence level:

$$\text{VaR}_P = \sigma_P \alpha \sqrt{x}$$

The VaR for a portfolio increases as the correlation between assets increase. The Basel Committee on Banking Supervision requires banks to hold capital for assets in the trading book of at least three times greater than 10-day VaR (i.e., VaR capital charge = 3 × 10-day VaR).

LO 6.5

The covariance matrix of assets is an important input for value at risk (VaR) and expected shortfall (ES). These risk management tools are sensitive to changes in correlation.

A lower default correlation is associated with greater diversification of credit risk. Creditors benefit by diversifying exposure across industries to lower the default correlations of debtors. The default term structure increases with time to maturity for most investment grade bonds. The probability of default is higher in the immediate time horizon for non-investment grade bonds.

LO 6.6

Systemic risk refers to the potential risk of a collapse of the entire financial system. The severity of correlation risk is even greater during a systemic crisis considering the higher correlations of U.S. equities with bonds and international equities.

Changes in the concentration risk, which is measured by the concentration ratio, are directly related to changes in default correlations. A lower concentration ratio and lower correlation coefficient both reduce the joint probability of default.

Concept Checkers

1. Suppose an individual buys a correlation swap with a fixed correlation of 0.2 and a notional value of $1 million for one year. The realized pairwise correlations of the daily log returns at maturity for three assets are $\rho_{2,1} = 0.7$, $\rho_{3,1} = 0.2$, and $\rho_{3,2} = 0.3$. What is the correlation swap buyer's payoff at maturity?
   A. $100,000.
   B. $200,000.
   C. $300,000.
   D. $400,000.

2. Suppose a financial institution has a two-asset portfolio with $7 million in asset A and $5 million in asset B. The portfolio correlation is 0.4, and the daily standard deviation of returns for asset A and B are 2% and 1%, respectively. What is the 10-day value at risk (VaR) of this portfolio at a 99% confidence level ($\alpha = 2.33$)?
   A. $1.226 million.
   B. $1.670 million.
   C. $2.810 million.
   D. $3.243 million.

3. In May of 2005, several large hedge funds had speculative positions in the collateralized debt obligations (CDOs) tranches. These hedge funds were forced into bankruptcy due to the lack of understanding of correlations across tranches. Which of the following statements best describes the positions held by hedge funds at this time and the role of changing correlations? Hedge funds held a:
   A. long equity tranche and short mezzanine tranche when the correlations in both tranches decreased.
   B. short equity tranche and long mezzanine tranche when the correlations in both tranches increased.
   C. short senior tranche and long mezzanine tranche when the correlation in the mezzanine tranche increased.
   D. long mezzanine tranche and short equity tranche when the correlation in the mezzanine tranche decreased.

4. Suppose a creditor makes a $4 million loan to company X and a $4 million loan to company Y. Based on historical information of companies in this industry, companies X and Y each have a 7% default probability and a default correlation coefficient of 0.6. The expected loss for this creditor under the worst case scenario assuming loss given default is 100% is closest to:
   A. $280,150.
   B. $351,680.
   C. $439,600.
   D. $560,430.

5. The relationship of correlation risk to credit risk is an important area of concern for risk managers. Which of the following statements regarding default probabilities and default correlations is incorrect?
   A. Creditors benefit by diversifying exposure across industries to lower the default correlations of debtors.
   B. The default term structure increases with time to maturity for most investment grade bonds.
   C. The probability of default is higher in the long-term time horizon for non-investment grade bonds.
   D. Changes in the concentration ratio are directly related to changes in default correlations.

Concept Checker Answers

1. **B** First, calculate the realized correlation as follows:

$$\rho_{\text{realized}} = \frac{2}{3^2 - 3} \times (0.7 + 0.2 + 0.3) = 0.4$$

The payoff for the correlation buyer is then calculated as:

$1,000,000 × (0.4 – 0.2) = $200,000

2. **A** The first step in solving for the 10-day VaR requires calculating the covariance matrix.

$$\text{cov}_{11} = \sigma_1^2 = 0.02^2 = 0.0004$$
$$\text{cov}_{22} = \sigma_2^2 = 0.01^2 = 0.0001$$
$$\text{cov}_{12} = \rho_{12} \times \sigma_1 \times \sigma_2 = 0.4 \times 0.02 \times 0.01 = 0.00008$$

Thus, the covariance matrix, C, can be represented as:

$$\begin{pmatrix} 0.0004 & 0.00008 \\ 0.00008 & 0.0001 \end{pmatrix}$$

Next, the standard deviation of the portfolio, σ_P, is determined as follows:

Step 1: Compute $\beta_h \times C$:

$$\begin{bmatrix} 7 & 5 \end{bmatrix} \begin{pmatrix} 0.0004 & 0.00008 \\ 0.00008 & 0.0001 \end{pmatrix}$$
$$= \begin{bmatrix} (7 \times 0.0004) + (5 \times 0.00008) & (7 \times 0.00008) + (5 \times 0.0001) \end{bmatrix}$$
$$= \begin{bmatrix} 0.0032 & 0.00106 \end{bmatrix}$$

Step 2: Compute $(\beta_h \times C) \times \beta_v$:

$$\begin{bmatrix} 0.0032 & 0.00106 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$
$$= (0.0032 \times 7) + (0.00106 \times 5) = 0.0277$$

Step 3: Compute σ_P:

$$\sigma_P = \sqrt{\beta_h \times C \times \beta_v} = \sqrt{0.0277} = 0.1664 \text{ or } 16.64\%$$

The 10-day portfolio VaR (in millions) at the 99% confidence level is then computed as:

$$\text{VaR}_P = \sigma_P \alpha \sqrt{x} = 0.1664 \times 2.33 \times \sqrt{10} = \$1.226 \text{ million}$$

3. **D** A number of large hedge funds were short on the CDO equity tranche and long on the CDO mezzanine tranche. Following the change in bond ratings for Ford and General Motors, the equity tranche spread increased dramatically. This caused losses on the short equity tranche position. At the same time, the correlation decreased for CDOs in the mezzanine tranche, which led to losses in the mezzanine tranche.

4. **B** The worst case scenario is the joint probability that both loans default at the same time. The joint probability of default is computed as:

$$P(AB) = 0.6\sqrt{0.07(0.93) \times 0.07(0.93)} + 0.07 \times 0.07$$
$$= 0.6\sqrt{0.00424} + 0.0049 = 0.04396$$

Thus, the expected loss for the worst case scenario for the creditor is:

EL = 0.04396 × \$8,000,000 = \$351,680

5. **C** The probability of default is higher in the *immediate* time horizon for non-investment grade bonds. The probability of default decreases over time if the company survives the near-term distressed situation.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Empirical Properties of Correlation: How Do Correlations Behave in the Real World?

Topic 7

Exam Focus

This topic examines how equity correlations and correlation volatility change during different economic states. It also discusses how to use a standard regression model to estimate the mean reversion rate and autocorrelation. For the exam, be able to calculate the mean reversion rate and be prepared to discuss and contrast the nature of correlations and correlation volatility for equity, bond, and default correlations. Also, be prepared to discuss the best fit distribution for these three types of correlation distributions.

Correlations During Different Economic States

LO 7.1: Describe how equity correlations and correlation volatilities behave throughout various economic states.

The recent financial crisis of 2007 provided new information on how correlation changes during different economic states. From 1972 to 2012, an empirical investigation on correlations of the 30 common stocks of the Dow Jones Industrial Average (Dow) was conducted. The correlation statistic was used to create a 30 × 30 correlation matrix for each stock in the Dow every month. This required 900 correlation calculations (30 × 30 = 900). There were 490 months in the study, so 441,000 monthly correlations were computed (900 × 490 = 441,000). However, the correlations of each stock with itself were eliminated from the study resulting in a total of 426,300 monthly correlations (441,000 – 30 × 490 = 426,300).

The average correlation values were compared for three states of the U.S. economy based on gross domestic product (GDP) growth rates. The state of the economy was defined as an expansionary period when GDP was greater than 3.5%, a normal economic period when GDP was between 0% and 3.5%, and a recession when there were two consecutive quarters of negative growth rates. Based on these definitions, from 1972 to 2012 there were six recessions, five expansionary periods, and five normal periods.

The average monthly correlation and correlation volatilities were then compared for each state of the economy. Correlation levels during a recession, normal period, and expansionary period were 37.0%, 32.7%, and 27.5%, respectively. Thus, as expected, correlations were highest during recessions when common stocks in equity markets tend to go down together. The low correlation levels during an expansionary period suggest common stock

valuations are determined more on industry and company-specific information rather than macroeconomic factors.

The correlation volatilities during a recession, normal period, and expansionary period were 80.5%, 83.4%, and 71.2%, respectively. These results may seem a little surprising at first as one may expect volatilities are highest during a recession. However, there is perhaps slightly more uncertainty in a normal economy regarding the overall direction of the stock market. In other words, investors expect stocks to go down during a recession and up during an expansionary period, but they are less certain of direction during normal times, which results in higher correlation volatility.

Professor Note: The main lesson from this portion of the study is that risk managers should be cognizant of high correlation and correlation volatility levels during recessions and times of extreme economic distress when calibrating risk management models.

Mean Reversion and Autocorrelation

LO 7.2: Calculate a mean reversion rate using standard regression and calculate the corresponding autocorrelation.

Mean reversion implies that over time, variables or returns regress back to the mean or average return. Empirical studies reveal evidence that bond values, interest rates, credit spreads, stock returns, volatility, and other variables are mean reverting. For example, during a recession, demand for capital is low. Therefore, interest rates are lowered to encourage investment in the economy. Then, as the economy picks up, demand for capital increases and, at some point, interest rates will rise. If interest rates are too high, demand for capital decreases and interest rates decrease and approach the long-run average. The level of interest rates is also a function of monetary and fiscal policy and not just supply and demand levels of capital.

Mean reversion is statistically defined as a negative relationship between the change in a variable over time, $S_t - S_{t-1}$, and the variable in the previous period, S_{t-1}:

$$\frac{\partial(S_t - S_{t-1})}{\partial S_{t-1}}$$

In this equation, S_t is the value of the variable at time period t, S_{t-1} is the value of the variable in the previous period, and ∂ is a partial derivative coefficient. Mean reversion exists when S_{t-1} increases (decreases) by a small amount causing $S_t - S_{t-1}$ to decrease (increase) by a small amount. For example, if S_{t-1} increases and is high at time period $t - 1$, then mean reversion causes the next value at S_t to reverse and decrease toward the long-run average or mean value. The **mean reversion rate** is the degree of the attraction back to the mean and is also referred to as the speed or gravity of mean reversion. The mean reversion rate, a, is expressed as follows:

$$S_t - S_{t-1} = a(\mu - S_{t-1})\Delta t + \sigma_S \varepsilon \sqrt{\Delta t}$$

If we are only concerned with measuring mean reversion, we can ignore the last term, $\sigma_S \varepsilon \sqrt{\Delta t}$, which is the stochastic part of the equation requiring random samples from a distribution over time. By ignoring the last term and assuming $\Delta t = 1$, the mean reversion rate equation simplifies to:

$$S_t - S_{t-1} = a(\mu - S_{t-1})$$

Example: Calculating mean reversion

Suppose mean reversion exists for a variable with a value of 50 at time period $t - 1$. The long-run mean value, μ, is 80. What are the expected changes in value of the variable over the next period, $S_t - S_{t-1}$, if the mean reversion rate, a, is 0, 0.5, or 1.0?

Answer:

If the mean reversion rate is 0, there is no mean reversion and there is no expected change. If the mean reversion rate is 0.5, there is a 50% mean reversion and the expected change is 15 [i.e., 0.5 × (80 – 50)]. If the mean reversion rate is 1.0, there is 100% mean reversion and the expected change is 30 [i.e., 1.0 × (80 – 50)]. Thus, a stronger or faster mean reversion is expected with a higher mean reversion rate.

Standard regression analysis is one method used to estimate the mean reversion rate, a. We can think of the mean reversion rate equation in terms of a standard regression equation (i.e., $Y = \alpha + \beta X$) by applying the distributive property to reformulate the right side of the equation:

$$S_t - S_{t-1} = a\mu - aS_{t-1}$$

Thinking of this equation in terms of a standard regression implies the following terms in the regression equation:

$$S_t - S_{t-1} = Y;\ a\mu = \alpha;\ \text{and} - aS_{t-1} = \beta X$$

A regression is run where $S_t - S_{t-1}$ (i.e., the Y variable) is regressed with respect to S_{t-1} (i.e., the X variable). Thus, the β coefficient of the regression is equal to the negative of the mean reversion rate, a.

From the 1972 to 2012 study, the data resulted in the following regression equation:

$$Y = 0.27 - 0.78X$$

The beta coefficient of –0.78 implies a mean reversion rate of 78%. This is a relatively high mean reversion rate. Thus, if there is a large decrease (increase) from the mean correlation

for one month, the following month is expected to have a large increase (decrease) in correlation.

Example: Calculating expected correlation

Suppose that in October 2012, the average monthly correlation for all Dow stocks was 30% and the long-run correlation mean of Dow stocks was 35%. A risk manager runs a regression, and the regression output estimates the following regression relationship: Y = 0.273 – 0.78X. What is the expected correlation for November 2012 given the mean reversion rate estimated in the regression analysis? (Solve for S_t in the mean reversion rate equation.)

Answer:

There is a 5% difference from the October 2012 and long-run mean correlation (35% – 30% = 5%). The β coefficient in the regression relationship implies a mean reversion rate of 78%. The November 2012 correlation is expected to revert 78% of the difference back toward the mean. Thus, the expected correlation for November 2012 is 33.9%:

$$S_t = a(\mu - S_{t-1}) + S_{t-1}$$

$$S_t = 0.78(35\% - 30\%) + 0.3 = 0.339$$

Autocorrelation measures the degree that a current variable value is correlated to past values. Autocorrelation is often calculated using an *autoregressive conditional heteroskedasticity (ARCH) model* or a *generalized autoregressive conditional heteroskedasticity (GARCH) model*. An alternative approach to measuring autocorrelation is running a regression equation. In fact, autocorrelation has the exact opposite properties of mean reversion.

Mean reversion measures the tendency to pull away from the current value back to the long-run mean. Autocorrelation instead measures the persistence to pull toward more recent historical values. The mean reversion rate in the previous example was 78% for Dow stocks. Thus, the autocorrelation for a one-period lag is 22% for the same sample. The sum of the mean reversion rate and the one-period autocorrelation rate will always equal one (i.e., 78% + 22% = 100%).

Autocorrelation for a one-period lag is statistically defined as:

$$AC(\rho_t, \rho_{t-1}) = \frac{\text{cov}(\rho_t, \rho_{t-1})}{\sigma(\rho_t) \times \sigma(\rho_{t-1})}$$

The term $AC(\rho_t, \rho_{t-1})$ represents the autocorrelation of the correlation from time period t and the correlation from time period $t - 1$. For this example, the ρ_t term can represent the correlation matrix for Dow stocks on day t, and the ρ_{t-1} term can represent the correlation

matrix for Dow stocks on day $t - 1$. The covariance between the correlation measures, $\text{cov}(\rho_t, \rho_{t-1})$, is calculated the same way covariance is calculated for equity returns.

This autocorrelation equation was used to calculate the one-period lag autocorrelation of Dow stocks for the 1972 to 2012 time period, and the result was 22%, which is identical to subtracting the mean reversion rate from one. The study also used this equation to test autocorrelations for 1- to 10-day lag periods for Dow stocks. The highest autocorrelation of 26% was found using a two-day lag, which compares the time period t correlation with the $t - 2$ correlation (two months prior). The autocorrelation for longer lags decreased gradually to approximately 10% using a 10-day lag. It is common for autocorrelations to decay with longer time period lags.

Professor Note: The autocorrelation equation is exactly the same as the correlation coefficient. Correlation values for time period t *and* t – 1 *are used to determine the autocorrelation between the two correlations.*

Best-Fit Distributions for Correlations

LO 7.3: Identify the best-fit distribution for equity, bond, and default correlations.

Seventy-seven percent of the correlations between stocks listed on the Dow from 1972 to 2012 were positive. Three distribution fitting tests were used to determine the best fit for equity correlations. Based on the results of the Kolmogorov-Smirnov, Anderson-Darling, and chi-squared distribution fitting tests, the **Johnson SB distribution** (which has two shape parameters, one location parameter, and one scale parameter) provided the best fit for equity correlations. The Johnson SB distribution best fit was also robust with respect to testing different economic states for the time period in question. The normal, lognormal, and beta distributions provided a poor fit for equity correlations.

There were three mild recessions and three severe recessions from 1972 to 2012. The time periods for the mild recessions occurred in 1980, 1990 to 1991, and 2001. More severe recessions occurred from 1973 to 1974 and from 1981 to 1982. Both of these severe recessions were caused by huge increases in oil prices. The most severe recession for this time period occurred from 2007 to 2009 following the global financial crisis. The percentage change in correlation volatility prior to a recession was negative in every case except for the 1990 to 1991 recession. This is consistent with the findings discussed earlier where correlation volatility is low during expansionary periods that often occur prior to a recession.

An empirical investigation of 7,645 bond correlations found average correlations for bonds of 42%. Correlation volatility for bond correlations was 64%. Bond correlations were also found to exhibit properties of mean reversion, but the mean reversion rate was only 26%. The best fit distribution for bond correlations was found to be the **generalized extreme value** (GEV) **distribution**. However, the normal distribution is also a good fit for bond correlations.

A study of 4,655 default probability correlations revealed an average default correlation of 30%. Correlation volatility for default probability correlations was 88%. The mean

reversion rate for default probability correlations was 30%, which is closer to the 26% for bond correlations. However, the default probability correlation distribution was similar to equity distributions in that the Johnson SB distribution is the best fit for both distributions. Figure 1 summarizes the findings of the empirical correlation analysis.

Figure 1: Empirical Findings for Equity, Bond, and Default Correlations

Correlation Type	*Average Correlation*	*Correlation Volatility*	*Reversion Rate*	*Best Fit Distribution*
Equity	35%	80%	78%	Johnson SB
Bond	42%	64%	26%	Generalized Extreme Value
Default Probability	30%	88%	30%	Johnson SB

Key Concepts

LO 7.1

Risk managers should be cognizant that historical correlation levels for common stocks in the Dow are highest during recessions. Correlation volatility for Dow stocks is high during recessions but highest during normal economic periods.

LO 7.2

When a regression is run where $S_t - S_{t-1}$ (the *Y* variable) is regressed with respect to S_{t-1} (the *X* variable), the β coefficient of the regression is equal to the negative mean reversion rate, *a*.

Equity correlations show high mean reversion rates (78%) and low autocorrelations (22%). These two rates must sum to 100%. Bond correlations and default probability correlations show much lower mean reversion rates and higher autocorrelation rates.

LO 7.3

Equity correlation distributions and default probability correlation distributions are best fit with the Johnson SB distribution. Bond correlation distributions are best fit with the generalized extreme value distribution, but the normal distribution is also a good fit.

Concept Checkers

1. Suppose a risk manager examines the correlations and correlation volatility of stocks in the Dow Jones Industrial Average (Dow) for the period beginning in 1972 and ending in 2012. Expansionary periods are defined as periods where the U.S. gross domestic product (GDP) growth rate is greater than 3.5%, periods are normal when the GDP growth rates are between 0 and 3.5%, and recessions are periods with two consecutive negative GDP growth rates. Which of the following statements characterizes correlation and correlation volatilities for this sample? The risk manager will most likely find that:
   A. correlations and correlation volatility are highest for recessions.
   B. correlations and correlation volatility are highest for expansionary periods.
   C. correlations are highest for normal periods, and correlation volatility is highest for recessions.
   D. correlations are highest for recessions, and correlation volatility is highest for normal periods.

2. Suppose mean reversion exists for a variable with a value of 30 at time period $t - 1$. Assume that the long-run mean value for this variable is 40 and ignore the stochastic term included in most regressions of financial data. What is the expected change in value of the variable for the next period if the mean reversion rate is 0.4?
   A. –10.
   B. –4.
   C. 4.
   D. 10.

3. A risk manager uses the past 480 months of correlation data from the Dow Jones Industrial Average (Dow) to estimate the long-run mean correlation of common stocks and the mean reversion rate. Based on historical data, the long-run mean correlation of Dow stocks was 32%, and the regression output estimates the following regression relationship: $Y = 0.24 - 0.75X$. Suppose that in April 2014, the average monthly correlation for all Dow stocks was 36%. What is the expected correlation for May 2014 assuming the mean reversion rate estimated in the regression analysis?
   A. 32%.
   B. 33%.
   C. 35%.
   D. 37%.

4. A risk manager uses the past 480 months of correlation data from the Dow Jones Industrial Average (Dow) to estimate the long-run mean correlation of common stocks and the mean reversion rate. Based on this historical data, the long-run mean correlation of Dow stocks was 34%, and the regression output estimates the following regression relationship: $Y = 0.262 - 0.77X$. Suppose that in April 2014, the average monthly correlation for all Dow stocks was 33%. What is the estimated one-period autocorrelation for this time period based on the mean reversion rate estimated in the regression analysis?
   A. 23%.
   B. 26%.
   C. 30%.
   D. 33%.

5. In estimating correlation matrices, risk managers often assume an underlying distribution for the correlations. Which of the following statements most accurately describes the best fit distributions for equity correlation distributions, bond correlation distributions, and default probability correlation distributions? The best fit distribution for the equity, bond, and default probability correlation distributions, respectively are:
   A. lognormal, generalized extreme value, and normal.
   B. Johnson SB, generalized extreme value, and Johnson SB.
   C. beta, normal, and beta.
   D. Johnson SB, normal, and beta.

Concept Checker Answers

1. **D** Findings of an empirical study of monthly correlations of Dow stocks from 1972 to 2012 revealed the highest correlation levels for recessions and the highest correlation volatilities for normal periods. The correlation volatilities during a recession and normal period were 80.5% and 83.4%, respectively.

2. **C** The mean reversion rate, *a,* indicates the speed of the change or reversion back to the mean. If the mean reversion rate is 0.4 and the difference between the last variable and long-run mean is 10 (= 40 – 30), the expected change for the next period is 4 (i.e., 0.4 × 10 = 4).

3. **B** There is a –4% difference from the long-run mean correlation and April 2014 correlation (32% – 36% = –4%). The inverse of the β coefficient in the regression relationship implies a mean reversion rate of 75%. Thus, the expected correlation for May 2014 is 33.0%:

$$S_t = a(\mu - S_{t-1}) + S_{t-1}$$

$$S_t = 0.75(32\% - 36\%) + 0.36 = 0.33$$

4. **A** The autocorrelation for a one-period lag is 23% for the same sample. The sum of the mean reversion rate (77% given the beta coefficient of –0.77) and the one-period autocorrelation rate will always equal 100%.

5. **B** Equity correlation distributions and default probability correlation distributions are best fit with the Johnson SB distribution. Bond correlation distributions are best fit with the generalized extreme value distribution.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Statistical Correlation Models—Can We Apply Them to Finance?

Topic 8

Exam Focus

This topic addresses the limitations of financial models and popular statistical correlation measures such as the Pearson correlation measure, the Spearman rank correlation, and the Kendall τ. For the exam, understand that the major limitation of the Pearson correlation coefficient is that most financial variables have nonlinear relationships. Also, be able to discuss the limitations of ordinal correlation measures, such as Spearman's rank correlation and Kendall's τ. These nonparametric measures do not require assumptions about the underlying joint distributions of variables; however, applications of ordinal risk measures are limited to ordinal variables where only the rankings are important instead of actual numerical values.

Limitations of Financial Models

LO 8.1: Evaluate the limitations of financial modeling with respect to the model itself, calibration of the model, and the model's output.

Financial models are important tools to help individuals and institutions better understand the complexity of the financial world. Financial models always deal with uncertainty and are, therefore, only approximations of a very complex pricing system that is influenced by numerous dynamic factors. There are many different types of markets trading a variety of assets and financial products such as equities, bonds, structured products, derivatives, real estate, and exchange-traded funds. Data from multiple sources is then gathered to calibrate financial models.

Due to the complexity of the global financial system, it is important to recognize the limitations of financial models. Limitations arise in financial models as a result of inaccurate inputs, erroneous assumptions regarding asset variable distributions, and mathematical inconsistencies. Almost all financial models require *market valuations* as inputs. Unfortunately, these values are often determined by investors who do not always behave rationally. Therefore, asset values are sometimes random and may exhibit unexpected changes.

Financial models also require assumptions regarding the *underlying distribution* of the asset returns. Value at risk (VaR) models are used to estimate market risk, and these models often assume that asset returns follow a normal distribution. However, empirical studies actually find higher kurtosis in return distributions, which suggest a distribution with fatter tails than the normal distribution.

Another example of a shortcoming of financial models is illustrated with the Black-Scholes-Merton (BSM) option pricing model. The BSM option pricing model assumes strike prices have constant volatility. However, numerous empirical studies find higher volatility for out-of-the money options and a volatility skew in equity markets. Thus, option traders and risk managers often use a volatility smile (discussed in Topic 15) with higher volatilities for out-of-the money call and put options.

Financial models at times may fail to accurately measure risk due to *mathematical inconsistencies*. For example, regarding barrier options, when applying the BSM option pricing model to up-and-out calls and puts and down-and-out calls and puts, there are rare cases where the inputs make the model insensitive to changes in implied volatility and option maturity. This can occur when the knock-out strike price is equal to the strike price, and the interest rate equals the underlying asset return. Risk managers and traders need to be aware of the possibility of mathematical inconsistencies causing model risk that leads to incorrect pricing and the inability to properly hedge risk.

Limitations in the Calibration of Financial Models

Financial models calibrate parameter inputs to reflect current market values. These parameters are then used in financial models to estimate market values with limited or no pricing information. The choice of time period used to calibrate the parameter inputs for the model can have a big impact on the results. For example, during the 2007 to 2009 financial crisis, risk managers used volatility and correlation estimates from pre-crisis periods. This resulted in significantly underestimating the risk for value at risk (VaR), credit value at risk (CVaR), and collateralized debt obligation (CDO) models.

All financial models should be tested using scenarios of extreme economic conditions. This process is referred to as **stress testing**. For example, VaR estimates are calculated in the event of a systemic financial crisis or severe recession. In 2012, the Federal Reserve, under the guidelines of Basel III, required all financial institutions to use stress tests.

Limitations of Financial Model Outputs

Limitations of financial models became evident during the recent global financial crisis. Traders and risk managers used new copula correlation models to estimate values in collateralized debt obligation (CDO) models. The values of these structured products were linked to mortgages in a collapsing real estate market.

The copula correlation models failed for two reasons. First, the copula correlation models assumed a negative correlation between the equity and senior tranches of CDOs. However, during the crisis, the correlations for both tranches significantly increased causing losses for both. Second, the copula correlation models were calibrated using volatility and correlation estimates with data from time periods that had low risk, and correlations changed significantly during the crisis.

A major lesson learned from the global financial crisis is that copula models cannot be blindly trusted. There should always be an element of human judgment in assessing the risk associated with any financial model. This is especially true for extreme market conditions.

Statistical Correlation Measures

LO 8.2: Assess the Pearson correlation approach, Spearman's rank correlation, and Kendall's τ, and evaluate their limitations and usefulness in finance.

Pearson Correlation

The Pearson correlation coefficient is commonly used to measure the *linear relationship* between two variables. The Pearson correlation is defined by dividing covariance (cov_{XY}) by the product of the two assets' standard deviations ($\sigma_X \sigma_Y$).

$$\rho_{XY} = \frac{cov_{XY}}{\sigma_X \sigma_Y}$$

Covariance is a measure of how two assets move with each other over time. The Pearson correlation coefficient standardizes covariance by dividing it by the standard deviations of each asset. This is very convenient because the correlation coefficient is always between –1 and 1.

Covariance is calculated by finding the product of each asset's deviation from their respective mean return for each period. The products of the deviations for each period are then added together and divided by the number of observations less one for degrees of freedom.

$$cov_{XY} = \frac{\sum_{t=1}^{n}(X_t - \mu_X)(Y_t - \mu_Y)}{n-1}$$

There is a second methodology that is used for calculating the Pearson correlation coefficient if the data is drawn from random processes with unknown outcomes (e.g., rolling a die). The following equation defines covariance with expectation values. If *E(X)* and *E(Y)* are the expected values of variables *X* and *Y*, respectively, then the expected product of deviations from these expected values is computed as follows:

$$E\{[X - E(X)][Y - E(Y)]\} \text{ or } E(XY) - E(X)E(Y)$$

When using random sets of data, the correlation coefficient can be rewritten as:

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - (E(X))^2} \times \sqrt{E(Y^2) - (E(Y))^2}}$$

Because many financial variables have nonlinear relationships, the Pearson correlation coefficient is only an approximation of the nonlinear relationship between financial

variables. Thus, when applying the Pearson correlation coefficient in financial models, risk managers and investors need to be aware of the following five limitations:

1. The Pearson correlation coefficient measures the linear relationship between two variables, but financial relationships are often nonlinear.

2. A Pearson correlation of zero does not imply independence between the two variables. It simply means there is not a linear relationship between the variables. For example, the parabola relationship defined as $Y = X^2$ has a correlation coefficient of zero. There is, however, an obvious nonlinear relationship between variables *Y* and *X*.

3. When the joint distribution between variables is not elliptical, linear correlation measures do not have meaningful interpretations. Examples of common elliptical joint distributions are the multivariate normal distribution and the multivariate Student's *t*-distribution.

4. The Pearson correlation coefficient requires that the variance calculations of the variables *X* and *Y* are finite. In cases where kurtosis is very high, such as the Student's *t*-distribution, the variance could be infinite, so the Pearson correlation coefficient would be undefined.

5. The Pearson correlation coefficient is not meaningful if the data is transformed. For example, the correlation coefficient between two variables *X* and *Y* will be different than the correlation coefficient between ln(X) and ln(Y).

Spearman's Rank Correlation

Ordinal measures are based on the order of elements in data sets. Two examples of ordinal correlation measures are the Spearman rank correlation and the Kendall τ. The Spearman rank correlation is a nonparametric approach because no knowledge of the joint distribution of the variables is necessary. The calculation is based on the relationship of the ranked variables. The following equation defines the Spearman rank correlation coefficient where *n* is the number of observations for each variable, and d_i is the difference between the ranking for period *i*.

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}$$

The Spearman rank correlation coefficient is determined in three steps:

Step 1: Order the set pairs of variables *X* and *Y* with respect to the set *X*.
Step 2: Determine the ranks of X_i and Y_i for each time period *i*.
Step 3: Calculate the difference of the variable rankings and square the difference.

Example: Spearman's rank correlation

Calculate the Spearman rank correlation for the returns of stocks *X* and *Y* provided in Figure 1.

Figure 1: Returns for Stocks *X* and *Y*

Year	*X*	*Y*
2010	25.0%	–20.0%
2011	60.0%	40.0%
2012	–20.0%	10.0%
2013	40.0%	20.0%
2014	–10.0%	30.0%
Average	19.0%	16.0%

Answer:

The calculations for determining the Spearman rank correlation coefficient are shown in Figure 2. The first step involves ranking the returns for stock *X* from lowest to highest in the second column. The first column denotes the respective year for each return. The returns for stock *Y* are then listed for each respective year. The fourth and fifth columns rank the returns for variables *X* and *Y*. The differences between the rankings for each year are listed in column six. Lastly, the sum of squared differences in rankings is determined in column 7.

Figure 2: Ranking Returns for Stocks *X* and *Y*

Year	*X*	*Y*	*X Rank*	*Y Rank*	d_i	d_i^2
2012	–20.0%	10.0%	1	2	-1	1
2014	–10.0%	30.0%	2	4	-2	4
2010	25.0%	–20.0%	3	1	2	4
2013	40.0%	20.0%	4	3	1	1
2011	60.0%	40.0%	5	5	0	0
					Sum =	10

The Spearman rank correlation coefficient can then be determined as 0.5:

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} = 1 - \frac{6 \times 10}{5(25-1)} = 0.5$$

Kendall's τ

Kendall's τ is another ordinal correlation measure that is becoming more widely applied in financial models for ordinal variables such as credit ratings. Kendall's τ is also a nonparametric measure that does not require any assumptions regarding the joint probability distributions of variables. Both Spearman's rank correlation coefficient and Kendall's τ are similar to the Pearson correlation coefficient for ranked variables because perfectly correlated variables will have a coefficient of 1. The Kendall τ will be 1 if variable *Y* always increases with an increase in variable *X*. The numerical amount of the increase does not matter for two variables to be perfectly correlated. Therefore, for most cases, the Kendall τ and the Spearman rank correlation coefficients will be different.

The mathematical definition of Kendall's τ is provided as follows:

$$\tau = \frac{n_c - n_d}{n(n-1)/2}$$

In this equation, the number of concordant pairs is represented as n_c, and the number of discordant pairs is represented as n_d. A concordant pair of observations is when the rankings of two pairs are in agreement:

$$X_t < Y_t \text{ and } X_{t*} < Y_{t*} \text{ or } X_t > Y_t \text{ and } X_{t*} > Y_{t*} \text{ and } t \neq t*$$

A discordant pair of observations is when the rankings of two pairs are not in agreement:

$$X_t < Y_t \text{ and } X_{t*} > Y_{t*} \text{ or } X_t > Y_t \text{ and } X_{t*} < Y_{t*} \text{ and } t \neq t*$$

A pair of rankings is neither concordant nor discordant if the rankings are equal:

$$X_t = Y_t \text{ or } X_{t*} = Y_{t*}$$

The denominator in the Kendall τ equation computes the total number of pair combinations. For example, if there are six pairs of observations, there will be 15 combinations of pairs:

$$[n \times (n - 1)] / 2 = (6 \times 5) / 2 = 15$$

Example: Kendall's τ

Calculate the Kendall τ correlation coefficient for the stock returns of *X* and *Y* listed in Figure 3.

Figure 3: Ranked Returns for Stocks *X* and *Y*

Year	*X*	*Y*	*X Rank*	*Y Rank*
2012	–20.0%	10.0%	1	2
2014	–10.0%	30.0%	2	4
2010	25.0%	–20.0%	3	1
2013	40.0%	20.0%	4	3
2011	60.0%	40.0%	5	5

Answer:

Begin by comparing the rankings of *X* and *Y* stock returns in columns four and five of Figure 3. There are five pairs of observations, so there will be ten combinations. Figure 4 summarizes the pairs of rankings based on the stock returns for *X* and *Y*. There are two concordant pairs, four discordant pairs, and four pairs that are neither concordant nor discordant.

Figure 4: Categorizing Pairs of Stock *X* and *Y* Returns

Concordant Pairs	Discordant Pairs	Neither
{(1,2),(2,4)}	{(1,2),(3,1)}	{(1,2),(5,5)}
{(3,1),(4,3)}	{(1,2),(4,3)}	{(2,4),(5,5)}
	{(2,4),(3,1)}	{(3,1),(5,5)}
	{(2,4),(4,3)}	{(4,3),(5,5)}

Kendall's τ can then be determined as –0.2:

$$\tau = \frac{n_c - n_d}{n(n-1)/2} = \frac{2-4}{5(5-1)/2} = -0.2$$

Thus, the relationship between the stock returns of *X* and *Y* is slightly negative based on the Kendall τ correlation coefficient.

Limitations of Ordinal Risk Measures

Ordinal correlation measures based on ranking (i.e., Spearman's rank correlation and Kendall's τ) are implemented in copula correlation models to analyze the dependence of market prices and counterparty risk. Because ordinal numbers simply show the rank of observations, problems arise when ordinal measures are used for cardinal observations, which show the quantity, number, or value of observations.

Example: Impact of outliers on ordinal measures

Suppose we triple the returns of *X* in the previous example to show the impact of outliers. If outliers are important sources of information and financial variables are cardinal, what are the implications for ordinal correlation measures?

Answer:

Notice from Figure 5 that Spearman's rank correlation and Kendall's τ do not change with an increased probability of outliers. Thus, ordinal correlation measures are less sensitive to outliers, which are extremely important in VaR and stress test models during extreme economic conditions. Numerical values are not important for ordinal correlation measures where only the rankings matter. Thus, since outliers do not change the rankings, *ordinal measures underestimate risk by ignoring the impact of outliers.*

Figure 5: Ranking Returns with Outliers

Year	*3X*	*Y*	*3X Rank*	*Y Rank*	d_i	d_i^2
2012	–60.0%	10.0%	1	2	–1	1
2014	–30.0%	30.0%	2	4	–2	4
2010	75.0%	–20.0%	3	1	2	4
2013	120.0%	20.0%	4	3	1	1
2011	180.0%	40.0%	5	5	0	0
					Sum =	10

Another limitation of Kendall's τ occurs when there are a large number of pairs that are neither concordant nor discordant. In other words, the Kendall τ calculation can be distorted when there are only a few concordant and discordant pairs. For example, there were 4 out of 10 pairs that were neither concordant nor discordant in Figure 4. Thus, the Kendall τ calculation was based on only 6 out of 10, or 60%, of the observations.

Key Concepts

LO 8.1

Limitations of financial models arise due to inaccurate input values, erroneous underlying distribution assumptions, and mathematical inconsistencies.

Copula correlation models failed during the 2007–2009 financial crisis due to assumptions of a negative correlation between the equity and senior tranches in a collateralized debt obligation (CDO) structure and the calibration of correlation estimates with pre-crisis data.

LO 8.2

A major limitation of the Pearson correlation coefficient is that it measures linear relationships when most financial variables are nonlinear.

The Spearman rank correlation coefficient, where *n* is the number of observations for each variable and d_i is the difference between the ranking for period *i*, is computed as follows:

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}$$

The Kendall τ correlation coefficient, where the number of concordant pairs is represented as n_c and the number of discordant pairs is represented as n_d, is computed as follows:

$$\tau = \frac{n_c - n_d}{n(n-1)/2}$$

Spearman's rank and Kendall's τ correlation coefficients should not be used with cardinal financial variables because ordinal measures underestimate risk by ignoring the impact of outliers.

Concept Checkers

1. Kirk Rozenboom, FRM, uses the Black-Scholes-Merton (BSM) model to value options. Following the financial crisis of 2007–2009, he is more aware of the limitations of the BSM option pricing model. Which of the following statements best characterizes a major limitation of the BSM option pricing model?
   A. The BSM model assumes strike prices have nonconstant volatility.
   B. Option traders often use a volatility smile with lower volatilities for out-of-the-money call and put options when applying the BSM model.
   C. For up-and-out calls and puts, the BSM model is insensitive to changes in implied volatility when the knock-out strike price is equal to the strike price and the interest rate equals the underlying asset return.
   D. For down-and-out calls and puts, the BSM model is insensitive to changes in option maturity when the knock-out strike price is greater than the strike price and the interest rate is greater than the underlying asset return.

2. New copula correlation models were used by traders and risk managers during the 2007–2009 global financial crisis. This led to miscalculations in the underlying risk for structured products such as collateralized debt obligation (CDO) models. Which of the following statements least likely explains the failure of these new copula correlation models during the financial crisis?
   A. The copula correlation models assumed a negative correlation between the equity and senior tranches of CDOs.
   B. Correlations for equity tranches of CDOs increased during the financial crisis.
   C. The correlation copula models were calibrated with data from time periods that had low risk.
   D. Correlations for senior tranches of CDOs decreased during the financial crisis.

3. A risk manager gathers five years of historical returns to calculate the Spearman rank correlation coefficient for stocks X and Y. The stock returns for X and Y from 2010 to 2014 are as follows:

Year	*X*	*Y*
2010	5.0%	–10.0%
2011	50.0%	–5.0%
2012	–10.0%	20.0%
2013	–20.0%	40.0%
2014	30.0%	15.0%

   What is the Spearman rank correlation coefficient for the stock returns of X and Y?
   A. –0.7.
   B. –0.3.
   C. 0.3.
   D. 0.7.

4. A risk manager gathers five years of historical returns to calculate the Kendall τ correlation coefficient for stocks *X* and *Y*. The stock returns for *X* and *Y* from 2010 to 2014 are as follows:

Year	*X*	*Y*
2010	5.0%	–10.0%
2011	50.0%	–5.0%
2012	–10.0%	20.0%
2013	–20.0%	40.0%
2014	30.0%	15.0%

What is the Kendall τ correlation coefficient for the stock returns of *X* and *Y*?
A. –0.3.
B. –0.2.
C. 0.4.
D. 0.7.

5. A risk manager is using a copula correlation model to perform stress tests of financial risk during systemic economic crises. If the risk manager is concerned about extreme outliers, which of the following correlation coefficient measures should be used?
A. Kendall's τ correlation.
B. Ordinal correlation.
C. Pearson correlation.
D. Spearman's rank correlation.

Concept Checker Answers

1. **C** For up-and-out calls and puts and for down-and-out calls and puts, the BSM option pricing model is insensitive to changes in implied volatility when the knock-out strike price is equal to the strike price and the interest rate equals the underlying asset return. The BSM model assumes strike prices have a *constant* volatility, and option traders often use a volatility smile with *higher* volatilities for out-of-the-money call and put options.

2. **D** During the crisis, the correlations for both the equity and senior tranches of CDOs significantly increased causing losses in value for both.

3. **A** The following table illustrates the calculations used to determine the sum of squared ranking deviations:

Year	*X*	*Y*	*X Rank*	*Y Rank*	d_i	d_i^2
2013	–20.0%	40.0%	1	5	–4	16
2012	–10.0%	20.0%	2	4	–2	4
2010	5.0%	–10.0%	3	1	2	4
2014	30.0%	15.0%	4	3	1	1
2011	50.0%	–5.0%	5	2	3	9
					Sum =	34

Thus, the Spearman rank correlation coefficient is –0.7:

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)} = 1 - \frac{6 \times 34}{5(25 - 1)} = -0.7$$

4. **B** The following table provides the ranking of pairs with respect to *X*.

Year	*X*	*Y*	*X Rank*	*Y Rank*
2013	–20.0%	40.0%	1	5
2012	–10.0%	20.0%	2	4
2010	5.0%	–10.0%	3	1
2014	30.0%	15.0%	4	3
2011	50.0%	–5.0%	5	2

There are four concordant pairs and six discordant pairs shown as follows:

Concordant Pairs	*Discordant Pairs*
{(1,5),(2,4)}	{(1,5),(3,1)}
{(3,1),(4,3)}	{(1,5),(4,3)}
{(3,1),(5,2)}	{(1,5),(5,2)}
{(4,3),(5,2)}	{(2,4),(3,1)}
	{(2,4),(4,3)}
	{(2,4),(5,2)}

Thus, the Kendall τ correlation coefficient is –0.2:

$$\tau = \frac{n_c - n_d}{n(n-1)/2} = \frac{4-6}{5(5-1)/2} = -0.2$$

5. **C** The Pearson correlation coefficient is preferred to ordinal measures when outliers are a concern. Spearman's rank correlation and Kendall's τ are ordinal correlation coefficients that should not be used with cardinal financial variables because they underestimate risk by ignoring the impact of outliers.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Financial Correlation Modeling—Bottom-Up Approaches

Topic 9

Exam Focus

A copula is a joint multivariate distribution that describes how variables from marginal distributions come together. Copulas provide an alternative measure of dependence between random variables that is not subject to the same limitations as correlation in applications such as risk measurement. For the exam, understand how a correlation copula is created by mapping two or more unknown distributions to a known distribution that has well-defined properties. Also, know how the Gaussian copula is used to estimate joint probabilities of default for specific time periods and the default time for multiple assets. The material in this topic is relatively complex, so your focus here should be on gaining a general understanding of how a copula function is applied.

Copula Functions

LO 9.1: Explain the purpose of copula functions and the translation of the copula equation.

A **correlation copula** is created by converting two or more unknown distributions that may have unique shapes and mapping them to a known distribution with well-defined properties, such as the normal distribution. A copula creates a joint probability distribution between two or more variables while maintaining their individual marginal distributions. This is accomplished by mapping multiple distributions to a single multivariate distribution. For example, the following expression defines a **copula function**, C, that transforms an n-dimensional function on the interval [0,1] to a one-dimensional function.

$$C:[0,1]^n \rightarrow [0,1]$$

Suppose $G_i(u_i) \in [0,1]$ is a univariate, uniform distribution with $u_i = u_1, \ldots, u_n$, and $i \in N$ (i.e., i is an element of set N). A copula function, C, can then be defined as follows:

$$C[G_1(u_1),\ldots,G_n(u_n)] = F_n\left[F_1^{-1}(G_1(u_1)),\ldots,F_n^{-1}(G_n(u_n));\rho_F\right]$$

In this equation, $G_i(u_i)$ are the marginal distributions, F_n is the joint cumulative distribution function, F_1^{-1} is the inverse function of F_n, and ρ_F is the correlation matrix structure of the joint cumulative function F_n.

This copula function is translated as follows. Suppose there are n marginal distributions, $G_1(u_1)$ to $G_n(u_n)$. A copula function exists that maps the marginal distributions of $G_1(u_1)$ to $G_n(u_n)$ via $F_1^{-1}G_i(u_i)$ and allows for the joining of the separate values $F_1^{-1}G_i(u_i)$ to a single n-variate function $F_n\left[F_1^{-1}(G_1(u_1)),\ldots,F_n^{-1}(G_n(u_n))\right]$ that has a correlation matrix of ρ_F. Thus, this equation defines the process where unknown marginal distributions are mapped to a well-known distribution, such as the standard multivariate normal distribution.

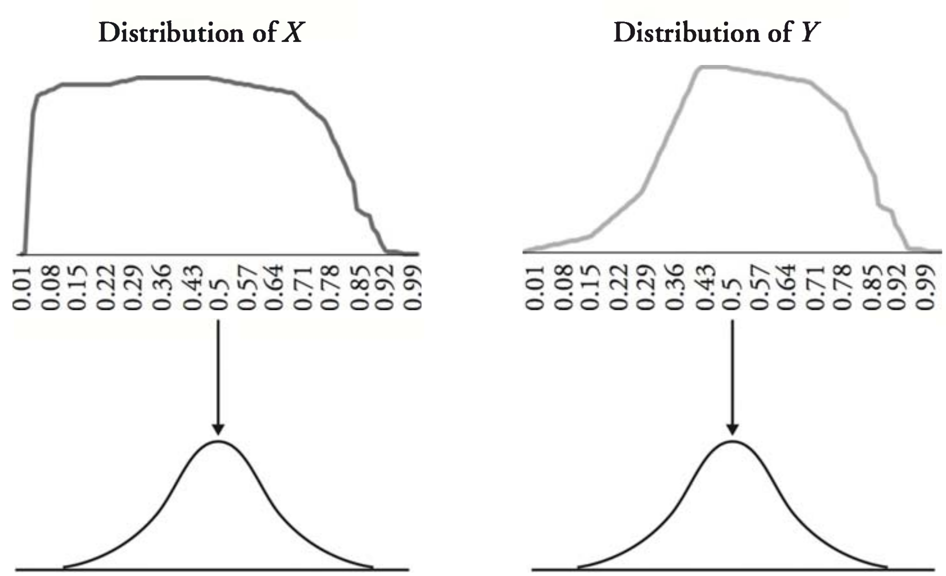
Gaussian Copula

LO 9.2: Describe the Gaussian copula and explain how to use it to derive the joint probability of default of two assets.

A **Gaussian copula** maps the marginal distribution of each variable to the standard normal distribution which, by definition, has a mean of zero and a standard deviation of one. The key property of a copula correlation model is preserving the original marginal distributions while defining a correlation between them. The mapping of each variable to the new distribution is done on percentile-to-percentile basis.

Figure 1 illustrates that the variables of two unknown distributions X and Y have unique marginal distributions. The observations of the unknown distributions are mapped to the standard normal distribution on a percentile-to-percentile basis to create a Gaussian copula.

Figure 1: Mapping a Gaussian Copula to the Standard Normal Distribution



For example, the 5th percentile observation for marginal distribution X is mapped to the 5th percentile point on the univariate standard normal distribution. When the 5th percentile is mapped, it will have a value of –1.645. This is repeated for each observation on

a percentile-to-percentile basis. Likewise, every observation on the marginal distribution of *Y* is mapped to the corresponding percentile on the univariate standard normal distribution. The new joint distribution is now a multivariate standard normal distribution.

Now a correlation structure can be defined between the two variables *X* and *Y*. The unique marginal distributions of *X* and *Y* are not well-behaved structures, and therefore, it is difficult to define a relationship between the two variables. However, the standard normal distribution is a well-behaved distribution. Therefore, a copula is a way to indirectly define a correlation relationship between two variables when it is not possible to directly define a correlation.

A Gaussian copula, C_G, is defined in the following expression for an *n*-variate example. The joint standard multivariate normal distribution is denoted as M_n. The inverse of the univariate standard normal distribution is denoted as N_1^{-1}. The notation ρ_M denotes the n × n correlation matrix for the joint standard multivariate normal distribution M_n.

$$C_G\left[G_1(u_1),\ldots,G_n(u_n)\right] = M_n\left[N_1^{-1}(G_1(u_1)),\ldots,N_n^{-1}(G_n(u_n));\rho_M\right]$$

In finance, the Gaussian copula is a common approach for measuring default risk. The approach can be transformed to define the **Gaussian default time copula**, C_{GD}, in the following expression:

$$C_{GD}\left[Q_i(t),\ldots,Q_n(t)\right] = M_n\left[N_1^{-1}(Q_1(t)),\ldots,N_n^{-1}(Q_n(t));\rho_M\right]$$

Marginal distributions of cumulative default probabilities, *Q(t)*, for assets $i = 1$ to *n* for fixed time periods *t* are mapped to the single *n*-variate standard normal distribution M_n with a correlation structure of ρ_M. The term $N_1^{-1}(Q_1(t))$ maps each individual cumulative default probability for asset *i* for time period *t* on a percentile-to-percentile basis to the standard normal distribution.

Example: Applying a Gaussian copula

Suppose a risk manager owns two non-investment grade assets. Figure 2 lists the default probabilities for the next five years for companies B and C that have B and C credit ratings, respectively. How can a Gaussian copula be constructed to estimate the joint default probability, *Q*, of these two companies in the next year, assuming a one-year Gaussian default correlation of 0.4?

Figure 2: Default Probabilities of Companies B and C

Time, t	*B Default Probability*	*C Default Probability*
1	0.065	0.238
2	0.081	0.152
3	0.072	0.113
4	0.064	0.092
5	0.059	0.072

Professor's Note: Non-investment grade companies have a higher probability of default in the near term during the company crisis state. If the company survives past the near term crisis, the probability of default will go down over time.

Answer:

In this example, there are only two companies, B and C. Thus, a bivariate standard normal distribution, M_2, with a default correlation coefficient of ρ can be applied. With two companies, only a single correlation coefficient is required, and not a correlation matrix of ρ_M.

$$C_{GD}\left[Q_B(t), Q_C(t)\right] = M_2\left[N^{-1}(Q_B(t)), N^{-1}(Q_C(t)); \rho\right]$$

Figure 3 illustrates the percentile-to-percentile mapping of cumulative default probabilities for each company to the standard normal distribution.

Figure 3: Mapping Cumulative Default Probabilities to Standard Normal Distribution

Time, t	*B Default Probability*	$Q_B(t)$	$N^{-1}(Q_B(t))$	*C Default Probability*	$Q_C(t)$	$N^{-1}(Q_C(t))$
1	0.065	0.065	–1.513	0.238	0.238	–0.712
2	0.081	0.146	–1.053	0.152	0.390	–0.279
3	0.072	0.218	–0.779	0.113	0.503	0.008
4	0.064	0.282	–0.577	0.092	0.595	0.241
5	0.059	0.341	–0.409	0.072	0.667	0.432

Columns 3 and 6 represent the cumulative default probabilities $Q_B(t)$ and $Q_C(t)$ for companies B and C, respectively. The values in columns 4 and 7 map the respective cumulative default probabilities, $Q_B(t)$ and $Q_C(t)$, to the standard normal distribution via $N^{-1}(Q(t))$. The values for the standard normal distribution are determined using the Microsoft Excel® function =NORMSINV(Q(t)) or the MATLAB® function =NORMINV(Q(t)). This process was illustrated graphically in Figure 1.

The joint probability of both Company B and Company C defaulting within one year is calculated as:

$$Q(t_B \leq 1 \cap t_C \leq 1) \equiv M(X_B \leq -1.513 \cap X_C \leq -0.712, \rho = 0.4) = 3.4\%$$

Professor's Note: You will not be asked to calculate the percentiles for mapping to the standard normal distribution because it requires the use of Microsoft Excel® or MATLAB®. In addition, you will not be asked to calculate the joint probability of default for a bivariate normal distribution due to its complexity.

Correlated Default Time

LO 9.3: Summarize the process of finding the default time of an asset correlated to all other assets in a portfolio using the Gaussian copula.

When a Gaussian copula is used to derive the default time relationship for more than two assets, a **Cholesky decomposition** is used to derive a sample $M_n(\bullet)$ from a multivariate copula $M_n(\bullet) \in [0,1]$. The default correlations of the sample are determined by the default correlation matrix ρ_M for the *n*-variate standard normal distribution, M_n.

The first step is to equate the sample $M_n(\bullet)$ to the cumulative individual default probability, *Q*, for asset *i* at time τ using the following equation. This is accomplished using Microsoft Excel® or a Newton-Raphson search procedure.

$$M_n(\bullet) = Q_i(\tau_i)$$

Next, the random samples are repeatedly drawn from the *n*-variate standard normal distribution $M_n(\bullet)$ to determine the expected default time using the Gaussian copula.

Random samples are drawn to estimate the default times, because there is no closed form solution for this equation.

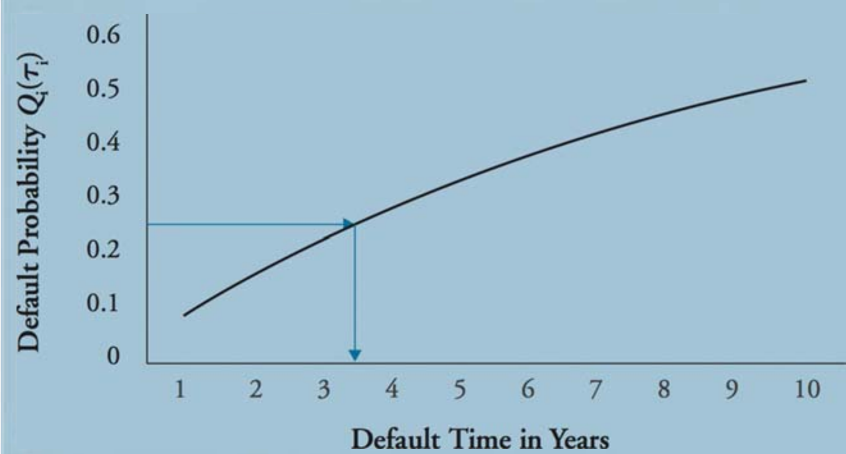
Example: Estimating default time

Illustrate how a risk manager estimates the expected default time of asset *i* using an *n*-variate Gaussian copula.

Answer:

Suppose a risk manager draws a 25% cumulative default probability for asset *i* from a random *n*-variate standard normal distribution, $M_n(\bullet)$. The *n*-variate standard normal distribution includes a default correlation matrix, ρ_M, that has the default correlations of asset *i* with all *n* assets. Figure 4 illustrates how to equate this 25% with the market determined cumulative individual default probability $Q_i(\tau_i)$. Suppose the first random sample equates to a default time τ of 3.5 years. This process is then repeated 100,000 times to estimate the default time of asset *i*.

Figure 4: Mapping Default Time for a Random Sample



Key Concepts

LO 9.1

The general equation for correlation copula, C, is defined as:

$$C[G_1(u_1),\ldots,G_n(u_n)] = F_n\left[F_1^{-1}(G_1(u_1)),\ldots,F_n^{-1}(G_n(u_n));\rho_F\right]$$

The notation for this copula equation is translated as: $G_i(u_i)$ are marginal distributions, F_n is the joint cumulative distribution function, F_1^{-1} is the inverse function of F_n, and ρ_F is the correlation matrix structure of the joint cumulative function F_n.

LO 9.2

The Gaussian default time copula is defined as:

$$C_{GD}[Q_i(t),\ldots,Q_n(t)] = M_n\left[N_1^{-1}(Q_1(t)),\ldots,N_n^{-1}(Q_n(t));\rho_M\right]$$

Marginal distributions of cumulative default probabilities, $Q(t)$, for assets i = 1 to n for fixed time periods t are mapped to the single n-variate standard normal distribution, M_n, with a correlation structure of ρ_M.

The Gaussian copula for the bivariate standard normal distribution, M_2, for two assets with a default correlation coefficient of ρ is defined as:

$$C_{GD}[Q_B(t),Q_C(t)] = M_2\left[N^{-1}(Q_B(t)),N^{-1}(Q_C(t));\rho\right]$$

LO 9.3

Random samples are drawn from an n-variate standard normal distribution sample, $M_n(\bullet)$, to estimate expected default times using the Gaussian copula:

$$M_n(\bullet) = Q_i(\tau_i)$$

Concept Checkers

1. Suppose a risk manager creates a copula function, C, defined by the equation:

$$C[G_1(u_1),...,G_n(u_n)] = F_n\left[F_1^{-1}(G_1(u_1)),...,F_n^{-1}(G_n(u_n));\rho_F\right]$$

Which of the following statements does not accurately describe this copula function?

A. $G_i(u_i)$ are standard normal univariate distributions.
B. F_n is the joint cumulative distribution function.
C. F_1^{-1} is the inverse function of F_n that is used in the mapping process.
D. ρ_F is the correlation matrix structure of the joint cumulative function F_n.

2. Which of the following statements best describes a Gaussian copula?

A. A major disadvantage of a Gaussian copula model is the transformation of the original marginal distributions in order to define the correlation matrix.
B. The mapping of each variable to the new distribution is done by defining a mathematical relationship between marginal and unknown distributions.
C. A Gaussian copula maps the marginal distribution of each variable to the standard normal distribution.
D. A Gaussian copula is seldom used in financial models because ordinal numbers are required.

3. A Gaussian copula is constructed to estimate the joint default probability of two assets within a one-year time period. Which of the following statements regarding this type of copula is incorrect?

A. This copula requires that the respective cumulative default probabilities are mapped to a bivariate standard normal distribution.
B. This copula defines the relationship between the variables using a default correlation matrix, ρ_M.
C. The term $N_1^{-1}(Q_1(t))$ maps each individual cumulative default probability for asset i for time period t on a percentile-to-percentile basis.
D. This copula is a common approach used in finance to estimate joint default probabilities.

4. A risk manager is trying to estimate the default time for asset i based on the default correlation copula of asset i to n assets. Which of the following equations best defines the process that the risk manager should use to generate and map random samples to estimate the default time?

A. $C_{GD}[Q_B(t),Q_C(t)] = M_2\left[N^{-1}(Q_B(t)),N^{-1}(Q_C(t));\rho\right]$
B. $C[G_1(u_1),...,G_n(u_n)] = F_n\left[F_1^{-1}(G_1(u_1)),...,F_n^{-1}(G_n(u_n));\rho_F\right]$
C. $C_{GD}[Q_i(t),...,Q_n(t)] = M_n\left[N_1^{-1}(Q_1(t)),...,N_n^{-1}(Q_n(t));\rho_M\right]$
D. $M_n(\bullet) = Q_i(\tau_i)$

5. Suppose a risk manager owns two non-investment grade assets and has determined their individual default probabilities for the next five years. Which of the following equations best defines how a Gaussian copula is constructed by the risk manager to estimate the joint probability of these two companies defaulting within the next year, assuming a Gaussian default correlation of 0.35?

A. $C_{GD}[Q_B(t), Q_C(t)] = M_2\left[N^{-1}(Q_B(t)), N^{-1}(Q_C(t)); \rho\right]$

B. $C[G_1(u_1), \ldots, G_n(u_n)] = F_n\left[F_1^{-1}(G_1(u_1)), \ldots, F_n^{-1}(G_n(u_n)); \rho_F\right]$

C. $C_{GD}[Q_i(t), \ldots, Q_n(t)] = M_n\left[N_1^{-1}(Q_1(t)), \ldots, N_n^{-1}(Q_n(t)); \rho_M\right]$

D. $M_n(\bullet) = Q_i(\tau_i)$

Concept Checker Answers

1. **A** $G_i(u_i)$ are marginal distributions that do not have well-known distribution properties.

2. **C** Observations of the unknown marginal distributions are mapped to the standard normal distribution on a percentile-to-percentile basis to create a Gaussian copula.

3. **B** Because there are only two companies, only a single correlation coefficient is required and not a correlation matrix, ρ_M.

4. **D** The equation $M_n(\bullet) = Q_i(\tau_i)$ is used to repeatedly generate random drawings from the *n*-variate standard normal distribution to determine the expected default time using the Gaussian copula.

5. **A** Because there are only two assets, the risk manager should use this equation to define the bivariate standard normal distribution, M_2, with a single default correlation coefficient of ρ.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Empirical Approaches to Risk Metrics and Hedging

Topic 10

Exam Focus

This topic discusses how dollar value of a basis point (DV01)-style hedges can be improved. Regression-based hedges enhance DV01-style hedges by examining yield changes over time. Principal components analysis (PCA) greatly simplifies bond hedging techniques. For the exam, understand the drawbacks of a standard DV01-netural hedge, and know how to compute the face value of an offsetting position using DV01 and how to adjust this position using regression-based hedging techniques.

DV01-Neutral Hedge

LO 10.1: Explain the drawbacks to using a DV01-neutral hedge for a bond position.

A standard DV01-neutral hedge assumes that the yield on a bond and the yield on a hedging instrument rise and fall by the same number of basis points. However, a one-to-one relationship does not always exist in practice. For example, if a trader hedges a T-bond which uses a nominal yield with a treasury security indexed to inflation [i.e., Treasury Inflation Protected Security (TIPS)] which uses a real yield, the hedge will likely be imprecise when changes in yield occur. In general, more dispersion surrounds the change in the nominal yield for a given change in the real yield. Empirically, the nominal yield adjusts by more than one basis point for every basis point adjustment in the real yield.

DV01-style metrics and hedges focus on how rates change relative to one another. As mentioned, the presumption that yields on nominal bonds and TIPS change by the same amount is not very realistic. To improve this DV01-neutral hedge approach, we can apply regression analysis techniques. Using a regression hedge examines the volatility of historical rate differences and adjusts the DV01 hedge accordingly, based on historical volatility.

Regression Hedge

LO 10.2: Describe a regression hedge and explain how it can improve a standard DV01-neutral hedge.

A regression hedge takes DV01-style hedges and adjusts them for projected nominal yield changes compared to projected real yield changes. Least squares regression analysis, which is used for regression-based hedges, looks at the historical relationship between real and nominal yields.

The advantage of a regression framework is that it provides an estimate of a hedged portfolio's volatility. An investor can gauge the expected gain in advance and compare it to historical volatility to determine whether the hedged portfolio is an attractive investment.

For example, assume a relative value trade is established whereby a trader sells a U.S. Treasury bond and buys a U.S. TIPS (which makes inflation-adjusted payments) to hedge the T-bond. The initial spread between these two securities represents the current views on inflation. Over time, changes in yields on nominal bonds and TIPS do not track one-for-one. To illustrate this hedge, assume the following data for yields and DV01s of a TIPS and a T-bond. Also assume that the trader is selling 100 million of the T-bond.

Bond	*Yield (%)*	*DV01*
TIPS	1.325	0.084
T-Bond	3.475	0.068

If the trade was made DV01-neutral, which assumes that the yield on the TIPS and the nominal bond will increase/decrease by the same number of basis points, the trade will not earn a profit or sustain a loss. The calculation for the amount of TIPS to purchase to hedge the short nominal bond is as follows:

$$F^R \times \frac{0.084}{100} = 100M \times \frac{0.068}{100}$$

$$F^R = 100M \times \frac{0.068}{0.084} = \$80.95 \text{ million}$$

where:
F^R = face amount of the real yield bond

To improve this hedge, the trader gathers yield data over time and plots a regression line, whereby the real yield is the independent variable and the nominal yield is the dependent variable. To compensate for the dispersion in the change in the nominal yield for a given change in the real yield, the trader would adjust the DV01-neutral hedge.

Hedge Adjustment Factor

LO 10.3: Calculate the regression hedge adjustment factor, beta.

In order to profit from a hedge, we must assume variability in the spread between the real and nominal yields over time. As mentioned, least squares regression is conducted to analyze these changes. The alpha and beta coefficients of a least squares regression line will be determined by the line of best fit through historical yield data points.

$$\Delta y_t^N = \alpha + \beta \Delta y_t^R + \varepsilon_t$$

where:
Δy_t^N = changes in the nominal yield
Δy_t^R = changes in the real yield

Recall that alpha represents the intercept term and beta represents the slope of the data plot. If least squares estimation determines the yield beta to be 1.0198, then this means that over the sample period, the nominal yield increases by 1.0198 basis points for every basis point increase in real yields.

LO 10.4: Calculate the face value of an offsetting position needed to carry out a regression hedge.

Defining F^R and F^N as the face amounts of the real and nominal bonds, respectively, and their corresponding DV01s as DV01^R and DV01^N, a DV01 hedge is adjusted by the hedge adjustment factor, or beta, as follows:

$$F^R = F^N \times \left(\frac{\text{DV01}^N}{\text{DV01}^R} \right) \times \beta$$

Now that we have determined the variability between the nominal and real yields, the hedge can be adjusted by the hedge adjustment factor of 1.0198:

$$F^R = 100\text{M} \times \left(\frac{0.068}{0.084} \right) \times 1.0198 = \$82.55 \text{ million}$$

This regression hedge approach suggests that for every $100 million sold in T-bonds, we should buy $82.55 million in TIPS. This will account for hedging not only the size of the underlying instrument, but also differences between nominal and real yields over time.

Note that in our example, the beta was close to one, so the resulting regression hedge did not change much from the DV01-neutral hedge. The regression hedge approach assumes that the hedge coefficient, β, is constant over time. This of course is not always the case, so it is best to estimate the coefficient over different time periods and make comparisons.

Two other factors should be also considered in our analysis: (1) the R-squared (i.e., the coefficient of determination), and (2) the standard error of the regression (SER). The

R-squared gives the percentage of variation in nominal yields that is explained by real yields. The standard error of the regression is the standard deviation of the realized error terms in the regression.

Two-Variable Regression Hedge

LO 10.5: Calculate the face value of multiple offsetting swap positions needed to carry out a two-variable regression hedge.

Regression hedging can also be conducted with two independent variables. For example, assume a trader in euro interest rate swaps buys/receives the fixed rate in a relatively illiquid 20-year swap and wishes to hedge this interest rate exposure. In this case, a regression hedge with swaps of different maturities would be appropriate. Since it may be impractical to hedge this position by immediately selling 20-year swaps, the trader may choose to sell a combination of 10- and 30-year swaps.

The trader is thus relying on a two-variable regression model to approximate the relationship between changes in 20-year swap rates and changes in 10- and 30-year swap rates. The following regression equation describes this relationship:

$$\Delta y_t^{20} = \alpha + \beta^{10}\Delta y_t^{10} + \beta^{30}\Delta y_t^{30} + \varepsilon_t$$

Similar to the single-variable regression hedge, this hedge of the 20-year euro swap can be expressed in terms of risk weights, which are the beta coefficients in the above equation:

$$\frac{(-F^{10} \times DV01^{10})}{(F^{20} \times DV01^{20})} = \text{change in 10-year swap rate, } \beta^{10}$$

$$\frac{(-F^{30} \times DV01^{30})}{(F^{20} \times DV01^{20})} = \text{change in 30-year swap rate, } \beta^{30}$$

The trader next does an initial regression analysis using data on changes in the 10-, 20-, and 30-year euro swap rates for a five-year time period. Assume the regression output is as follows:

Number of observations	1281
R-squared	99.8%
Standard error	0.14

Regression Coefficients	Value	Standard Error
Alpha	–0.0014	0.0040
Change in 10-year swap rate	0.2221	0.0034
Change in 30-year swap rate	0.7765	0.0037

Given these regression results and an illiquid 20-year swap, the trader would hedge 22.21% of the 20-year swap DV01 with a 10-year swap and 77.65% of the 20-year swap DV01 with a 30-year swap. Because these weights sum to approximately one, the regression hedge DV01 will be very close to the 20-year swap DV01.

The two-variable approach will provide a better hedge (in terms of R-squared) compared to a single-variable approach. However, regression hedging is not an exact science. There are several cases in which simply doing a one-security DV01 hedge, or a two-variable hedge with arbitrary risk weights, is not appropriate (e.g., hedging during a financial crisis).

Level and Change Regressions

LO 10.6: Compare and contrast level and change regressions.

When setting up and establishing regression-based hedges, there are two schools of thought. Some regress changes in yields on changes in yields, as demonstrated previously, but an alternative approach is to regress yields on yields.

Using a single-variable approach, the formula for a change-on-change regression with dependent variable *y* and independent variable *x* is as follows:

$$\Delta y_t = \alpha + \beta \Delta x_t + \Delta \varepsilon_t$$

where:
$\Delta y_t = y_t - y_{t-1}$
$\Delta x_t = x_t - x_{t-1}$

Alternatively, the formula for a level-on-level regression is as follows:

$$y_t = \alpha + \beta x_t + \varepsilon_t$$

With both approaches, the estimated regression coefficients are unbiased and consistent; however, the error terms are unlikely to be independent of each other. Thus, since the error terms are correlated over time (i.e., *serially correlated)*, the estimated regression coefficients are not efficient. As a result, there is a third way to model the relationship between two bond yields (for some constant correlation < 1):

$$\varepsilon_t = \rho \varepsilon_{t-1} + v_t$$

This formula assumes that today's error term consists of some part of yesterday's error term, plus a new random fluctuation.

Principal Components Analysis

LO 10.7: Describe principal component analysis and explain how it is applied to constructing a hedging portfolio.

Regression analysis focuses on yield changes among a small number of bonds. Empirical approaches, such as principal components analysis (PCA), take a different approach by providing a single empirical description of term structure behavior, which can be applied

across all bonds. PCA attempts to explain all factor exposures using a small number of uncorrelated exposures which do an adequate job of capturing risk.

For example, if we consider the set of swap rates from 1 to 30 years, at annual maturities, the PCA approach creates 30 interest rate factors or components, and each factor describes a change in each of the 30 rates. This is in contrast to regression analysis, which looks at variances of rates and their pairwise correlations.

PCA sets up the 30 factors with the following properties:

1. The sum of the variances of the 30 principal components (PCs) equals the sum of the variances of the individual rates. The PCs thus capture the volatility of the set of rates.

2. The PCs are not correlated with each other.

3. Each PC is chosen to contain the highest possible variance, given the earlier PCs.

The advantage of this approach is that we only really need to describe the volatility and structure of the first three PCs since the sum of the variances of the first three PCs is a good approximation of the sum of the variances of all rates. Thus, the PCA approach creates three factors that capture similar data as a comprehensive matrix containing variances and covariances of all interest rate factors. Changes in 30 rates can now be expressed with changes in three factors, which is a much simpler approach.

Key Concepts

LO 10.1

A DV01-neutral hedge assumes the yield on a bond and the yield on a hedging instrument rise and fall by the same number of basis points. However, empirically, a nominal yield will adjust by more than one basis point for every basis point adjustment in a real yield.

LO 10.2

A regression hedge adjusts for the extra movement in the projected nominal yield changes compared to the projected real yield changes.

LO 10.3

Least squares regression is conducted to analyze the changes in historical yields between nominal and real bonds.

$$\Delta y_t^N = \alpha + \beta \Delta y_t^R + \varepsilon_t$$

LO 10.4

A DV01 hedge is adjusted by the hedge adjustment factor, or beta, as follows:

$$F^R = F^N \times \left(\frac{DV01^N}{DV01^R} \right) \times \beta$$

LO 10.5

Regression hedging can also be conducted with a two-variable regression model. The beta coefficients in the regression model represent risk weights, which are used to calculate the face value of multiple offsetting positions.

LO 10.6

The formula for a level-on-level regression with dependent variable *y* and independent variable *x* is as follows:

$$y_t = \alpha + \beta x_t + \varepsilon_t$$

The formula for a change-on-change regression is as follows:

$$y_t - y_{t-1} = \Delta y_t = \alpha + \beta \Delta x_t + \Delta \varepsilon_t$$

With both approaches, the error terms are unlikely to be independent of each other.

LO 10.7

Principal components analysis (PCA) provides a single empirical description of term structure behavior, which can be applied across all bonds. The advantage of this approach is that we only need to describe the volatility and structure of a small number of principal components, which approximate all movements in the term structure.

Concept Checkers

1. If a trader is creating a fixed income hedge, which hedging methodology would be least effective if the trader is concerned about the dispersion of the change in the nominal yield for a particular change in the real yield?
   A. One-variable regression hedge.
   B. DV01 hedge.
   C. Two-variable regression hedge.
   D. Principal components hedge.

2. Assume that a trader is making a relative value trade, selling a U.S. Treasury bond and correspondingly purchasing a U.S. TIPS. Based on the current spread between the two securities, the trader shorts $100 million of the nominal bond and purchases $89.8 million of TIPS. The trader then starts to question the amount of the hedge due to changes in yields on TIPS in relation to nominal bonds. He runs a regression and determines from the output that the nominal yield changes by 1.0274 basis points per basis point change in the real yield. Would the trader adjust the hedge, and if so, by how much?
   A. No.
   B. Yes, by $2.46 million (purchase additional TIPS).
   C. Yes, by $2.5 million (sell a portion of the TIPS).
   D. Yes, by $2.11 million (purchase additional TIPS).

3. What is a key advantage of using a regression hedge to fine tune a DV01 hedge?
   A. It assumes that term structure changes are driven by one factor.
   B. The proper hedge amount may be computed for any assumed change in the term structure.
   C. Bond price changes and returns can be estimated with proper measures of price sensitivity.
   D. It gives an estimate of the hedged portfolio's volatility over time.

4. What does the regression hedge assume about the hedge coefficient, beta?
   A. It moves in lockstep with real rates.
   B. It stays constant over time.
   C. It generally tracks nominal rates over time.
   D. It is volatile over time, similar to both real and nominal rates.

5. Traci York, FRM, is setting up a regression-based hedge and is trying to decide between a changes-in-yields-on-changes-in-yields approach versus a yields-on-yields approach. Which of the following is a correct statement concerning error terms in these two approaches?
   A. In both cases, the error terms are completely uncorrelated.
   B. With change-on-change, there is no correlation in error terms, while yield-on-yield error terms are completely correlated.
   C. Error terms are correlated over time with both approaches.
   D. With yield-on-yield, there is no correlation in error terms, while change-on-change error terms are completely correlated.

Concept Checker Answers

1. **B** The DV01 hedge assumes that the yield on the bond and the assumed hedging instruments rises and falls by the same number of basis points; so with a DV01 hedge, there is not much the trader can do to allow for dispersion between nominal and real yields.

2. **B** The trader would need to adjust the hedge as follows:

   \$89.8 million × 1.0274 = \$92.26 million

   Thus, the trader needs to purchase additional TIPS worth \$2.46 million.

3. **D** A key advantage of using a regression approach in setting up a hedge is that it automatically gives an estimate of the hedged portfolio's volatility.

4. **B** It should be pointed out that while it is true that the regression hedge assumes a constant beta, this is not a realistic assumption; thus, it is best to estimate beta over several time periods and compare accordingly.

5. **C** With the level-on-level approach, error terms are somewhat correlated over time, while with the change-on-change approach, the error terms are completely correlated. Thus, error terms are correlated over time with both approaches.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

The Science of Term Structure Models

Topic 11

Exam Focus

The emphasis of this topic is the pricing of interest rate derivative contracts using a risk-neutral binomial model. The pricing process for interest rate derivatives requires intensive calculations and is very tedious. However, the relationship becomes straightforward when it is modeled to support risk neutrality. Understand the concepts of backward induction and how the addition of time steps will increase the accuracy of any bond pricing model. Bonds with embedded options are also discussed in this topic. Be familiar with the price-yield relationship of both callable and putable bonds. This topic incorporates elements of material from the FRM Part I curriculum where you valued options with binomial trees.

Interest Rate Tree (Binomial) Model

LO 11.1: Calculate the expected discounted value of a zero-coupon security using a binomial tree.

LO 11.2: Construct and apply an arbitrage argument to price a call option on a zero-coupon security using replicating portfolios.

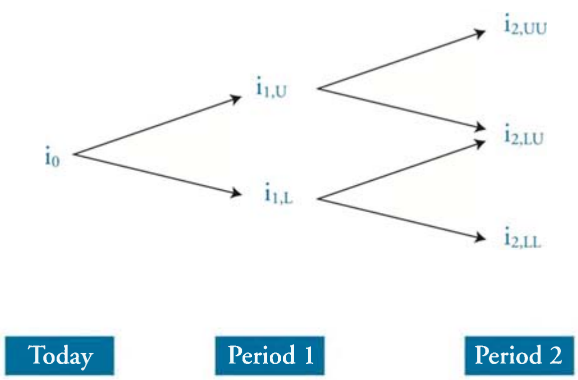
The binomial interest rate model is used throughout this topic to illustrate the issues that must be considered when valuing bonds with embedded options. A **binomial model** is a model that assumes that interest rates can take only one of two possible values in the next period.

This interest rate model makes assumptions about interest rate volatility, along with a set of paths that interest rates may follow over time. This set of possible interest rate paths is referred to as an **interest rate tree**.

Binomial Interest Rate Tree

The diagram in Figure 1 depicts a binomial interest rate tree.

Figure 1: 2-Period Binomial



To understand this 2-period binomial tree, consider the nodes indicated with the boxes in Figure 1. A node is a point in time when interest rates can take one of two possible paths—an upper path, *U*, or a lower path, *L*. Now consider the node on the right side of the diagram where the interest rate $i_{2,LU}$ appears. This is the rate that will occur if the initial rate, i_0, follows the lower path from node 0 to node 1 to become $i_{1,L}$, then follows the upper of the two possible paths to node 2, where it takes on the value $i_{2,LU}$. At the risk of stating the obvious, the upper path from a given node leads to a higher rate than the lower path. Notice also that an upward move followed by a downward move gets us to the same place on the tree as a down-then-up move, so $i_{2,LU} = i_{2,UL}$.

The interest rates at each node in this interest rate tree are 1-period forward rates corresponding to the nodal period. Beyond the root of the tree, there is more than one 1-period forward rate for each nodal period (i.e., at year 1, we have two 1-year forward rates, $i_{1,U}$ and $i_{1,L}$). The relationship among the rates associated with each individual nodal period is a function of the interest rate volatility assumption of the model being employed to generate the tree.

Constructing the Binomial Interest Rate Tree

The construction of an interest rate tree, binomial or otherwise, is a tedious process. In practice, the interest rate tree is usually generated using specialized computer software. There is one underlying rule governing the construction of an interest rate tree: *The values for on-the-run issues generated using an interest rate tree should prohibit arbitrage opportunities*. This means that the value of an on-the-run issue produced by the interest rate tree must equal its market price. It should be noted that in accomplishing this, the interest rate tree must maintain the interest rate volatility assumption of the underlying model.

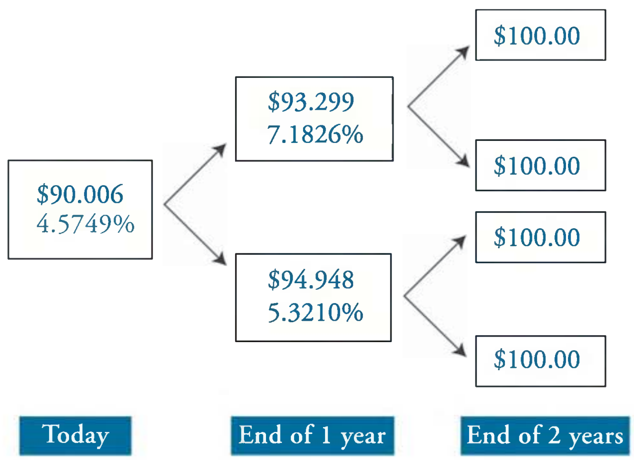
Valuing an Option-Free Bond With the Tree, Using Backward Induction

Backward induction refers to the process of valuing a bond using a binomial interest rate tree. The term "backward" is used because in order to determine the value of a bond at node 0, you need to know the values that the bond can take on at node 1. But to determine the values of the bond at node 1, you need to know the possible values of the bond at node 2,

and so on. Thus, for a bond that has N compounding periods, the current value of the bond is determined by computing the bond's possible values at period N and working "backward" to node 0.

Consider the binomial tree shown in Figure 2 for a $100 face value, zero-coupon bond, with two years remaining until maturity, and a market price of $90.006. Starting on the top line, the blocks at each node include the value of the bond and the 1-year forward rate at that node. For example, at the upper path of node 1, the price is $93.299, and the 1-year forward rate is 7.1826%.

Figure 2: Valuing a 2-Year, Zero-Coupon, Option-Free Bond



Know that *the value of a bond at a given node in a binomial tree is the **average** of the present values of the two possible values from the next period.* The appropriate discount rate is the forward rate associated with the node under analysis.

Example: Valuing an option-free bond

Assuming the bond's market price is $90.006, **demonstrate** that the tree in Figure 2 is arbitrage free using backward induction.

Answer:

Consider the value of the bond at the *upper* node for period 1, $V_{1,U}$:

$$V_{1,U} = \frac{(\$100 \times 0.5) + (\$100 \times 0.5)}{1.071826} = \$93.299$$

Similarly, the value of the bond at the *lower* node for period 1, $V_{1,L}$ is:

$$V_{1,L} = \frac{(\$100 \times 0.5) + (\$100 \times 0.5)}{1.053210} = \$94.948$$

Now **calculate** V_0, the current value of the bond at node 0:

$$V_0 = \frac{(\$93.299 \times 0.5) + (\$94.948 \times 0.5)}{1.045749} = \$90.006$$

Since the computed value of the bond equals the market price, the binomial tree is arbitrage free.

Professor's Note: When valuing bonds with coupon payments, you need to add the coupons to the bond prices at each node. For example, with a \$100 face value, 7% annual coupon bond, you would add the \$7 coupon to each price before computing present values. Valuing coupon-paying bonds with a binomial tree will be illustrated in LO 11.5.

LO 11.3: Define risk-neutral pricing and apply it to option pricing.

LO 11.4: Distinguish between true and risk-neutral probabilities, and apply this difference to interest rate drift.

Using the 0.5 probabilities for up and down states as shown in the previous example may not produce an expected discounted value that exactly matches the market price of the bond. This is because the 0.5 probabilities are the assumed **true probabilities** of price movements. In order to equate the discounted value using a binomial tree and the market price, we need to use what is known as **risk-neutral probabilities**. Any difference between the risk-neutral and true probabilities is referred to as the **interest rate drift**.

Using the Risk-Neutral Interest Rate Tree

There are actually two ways to compute bond and bond derivative values using a binomial model. These techniques are referred to as **risk-neutral pricing**.

- The first method is to start with spot and forward rates derived from the current yield curve and then *adjust the interest rates* on the paths of the tree so that the value derived from the model is equal to the current market price of an on-the-run bond (i.e., the tree is created to be "arbitrage free"). This is the method we used in the previous example. Once the interest rate tree is derived for an on-the-run bond, we can use it to price derivative securities on the bond by calculating the expected discounted value at each node using the real-world probabilities.
- The second method is to take the rates on the tree as given and then *adjust the probabilities* so that the value of the bond derived from the model is equal to its current market price. Once we derive these *risk-neutral probabilities*, we can use them to price derivative securities on the bond by once again calculating the expected discounted value at each node using the risk-neutral probabilities and working backward through the tree.

The value of the derivative is the same under either method.

LO 11.5: Explain how the principles of arbitrage pricing of derivatives on fixed income securities can be extended over multiple periods.

There are three basic steps to valuing an option on a fixed-income instrument using a binomial tree:

Step 1: Price the bond value at each node using the projected interest rates.
Step 2: Calculate the intrinsic value of the derivative at each node at maturity.
Step 3: Calculate the expected discounted value of the derivative at each node using the risk-neutral probabilities and working backward through the tree.

Note that the option cannot be properly priced using expected discounted values because the call option value depends on the path of interest rates over the life of the option. Incorporating the various interest rate paths will prohibit arbitrage from occurring.

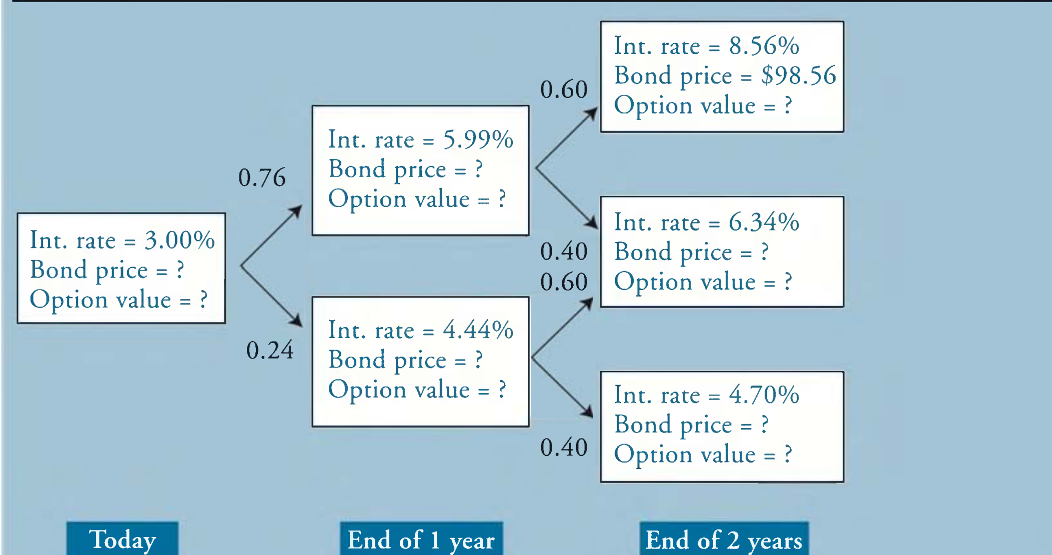
Example: Call option

Assume that you want to value a European call option with two years to expiration and a strike price of $100.00. The underlying is a 7%, annual-coupon bond with three years to maturity. Figure 3 represents the first two years of the binomial tree for valuing the underlying bond. Assume that the risk-neutral probability of an up move is 0.76 in year 1 and 0.60 in year 2.

Fill in the missing data in the binomial tree, and **calculate** the value of the European call option.

Professor's Note: Since the option is European, it can only be exercised at maturity.

Figure 3: Incomplete Binomial Tree for European Call Option on 3-Year, 7% Bond



Answer:

Step 1: Calculate the bond prices at each node using the backward induction methodology.

At the middle node in year 2, the price is \$100.62. You can calculate this by noting that at the end of year 2 the bond has one year left to maturity:

$$N = 1; I/Y = 6.34; PMT = 7; FV = 100; CPT \rightarrow PV = 100.62$$

At the bottom node in year 2, the price is \$102.20:

$$N = 1; I/Y = 4.70; PMT = 7; FV = 100; CPT \rightarrow PV = 102.20$$

At the top node in year 1, the price is \$100.37:

$$\frac{(\$105.56 \times 0.6) + (\$107.62 \times 0.4)}{1.0599} = \$100.37$$

At the bottom node in year 1, the price is \$103.65:

$$\frac{(\$107.62 \times 0.6) + (\$109.20 \times 0.4)}{1.0444} = \$103.65$$

Today, the price is \$105.01:

$$\frac{(\$107.37 \times 0.76) + (\$110.65 \times 0.24)}{1.03} = \$105.01$$

As shown here, the price at a given node is the expected discounted value of the cash flows associated with the two nodes that "feed" into that node. The discount rate that is applied is the prevailing interest rate at the given node. Note that since this is a European option, you really only need the bond prices at the maturity date of the option (end of year 2) if you are given the arbitrage-free interest rate tree. However, it's good practice to compute all the bond prices.

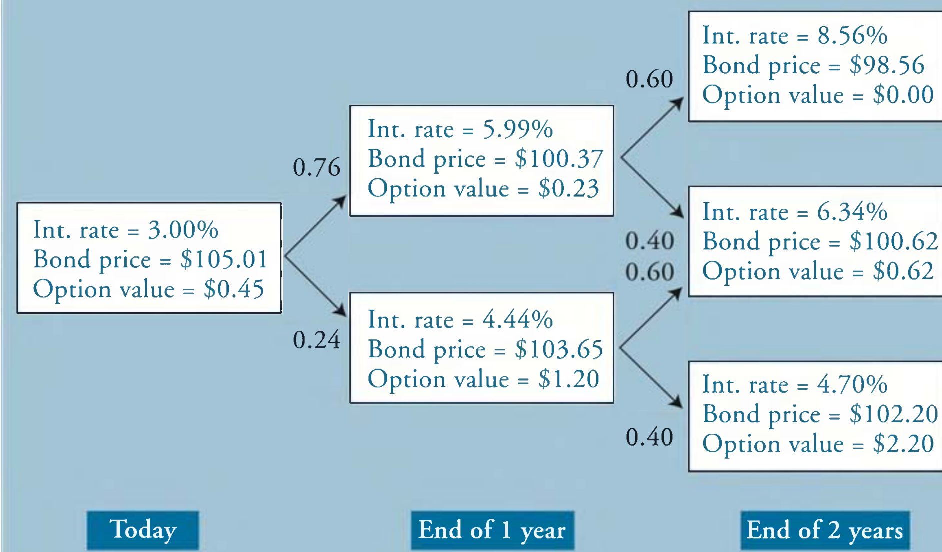
Step 2: Determine the intrinsic value of the option at maturity in each node. For example, the intrinsic value of the option at the bottom node at the end of year 2 is \$2.20 = \$102.20 – \$100.00. At the top node in year 2, the intrinsic value of the option is zero since the bond price is less than the call price.

Step 3: Using the backward induction methodology, calculate the option value at each node prior to expiration. For example, at the top node for year 1, the option price is \$0.23:

$$\frac{(\$0.00 \times 0.6) + (\$0.62 \times 0.4)}{1.0599} = \$0.23$$

Figure 4 shows the binomial tree with all values included.

Figure 4: Completed Binomial Tree for European Call Option on 3-Year, 7% Bond



The option value today is computed as:

$$\frac{(\$0.23 \times 0.76) + (\$1.20 \times 0.24)}{1.03} = \$0.45$$

Recombining and Nonrecombining

LO 11.7: Describe the rationale behind the use of recombining trees in option pricing.

In the previous example, the interest rate in the middle node of period two was the same (i.e., 6.34%) regardless of the path being up then down or down then up. This is known as a **recombining tree**. It may be the case, in a practical setting, that the up then down scenario produces a different rate than the down then up scenario. An example of this type of tree may result when any interest rate above a certain level (e.g., 3%) causes rates to move a fixed number of basis points, but any interest rate below that level causes rates to move at a pace that is below the up state's fixed amount. When rates move in this fashion, the movement process is known as **state-dependent volatility**, and it results in **nonrecombining trees**. From an economic standpoint, nonrecombining trees are appropriate; however, prices can be very difficult to calculate when the binomial tree is extended to multiple periods.

Constant Maturity Treasury Swap

LO 11.8: Calculate the value of a constant maturity Treasury swap, given an interest rate tree and the risk-neutral probabilities.

In addition to valuing options with binomial interest rate trees, we can also value other derivatives such as swaps. The following example calculates the price of a **constant maturity Treasury** (CMT) **swap**. A CMT swap is an agreement to swap a floating rate for a Treasury rate such as the 10-year rate.

Example: CMT swap

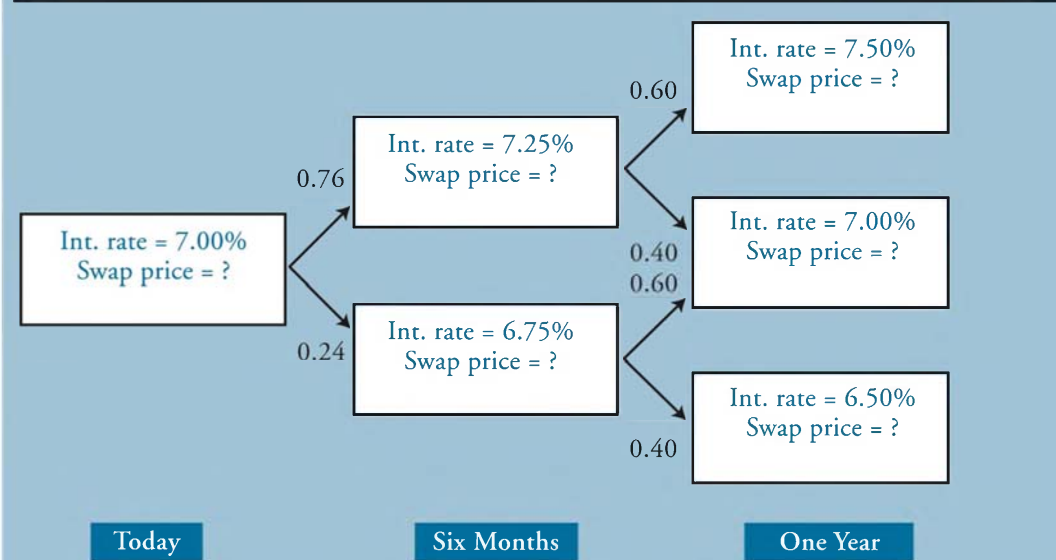
Assume that you want to value a constant maturity Treasury (CMT) swap. The swap pays the following every six months until maturity:

$$\left(\frac{\$1{,}000{,}000}{2}\right)\times\left(y_{CMT}-7\%\right)$$

y_{CMT} is a semiannually compounded yield, of a predetermined maturity, at the time of payment (y_{CMT} is equivalent to 6-month spot rates). Assume there is a 76% risk-neutral probability of an increase in the 6-month spot rate and a 60% risk-neutral probability of an increase in the 1-year spot rate.

Fill in the missing data in the binomial tree, and **calculate** the value of the swap.

Figure 5: Incomplete Binomial Tree for CMT Swap



Answer:

In six months, the top node and bottom node payoffs are, respectively:

$$\text{payoff}_{1,U} = \frac{\$1{,}000{,}000}{2} \times (7.25\% - 7.00\%) = \$1{,}250$$

$$\text{payoff}_{1,L} = \frac{\$1{,}000{,}000}{2} \times (6.75\% - 7.00\%) = -\$1{,}250$$

Similarly in one year, the top, middle, and bottom payoffs are, respectively:

$$\text{payoff}_{2,U} = \frac{\$1{,}000{,}000}{2} \times (7.50\% - 7.00\%) = \$2{,}500$$

$$\text{payoff}_{2,M} = \frac{\$1{,}000{,}000}{2} \times (7.00\% - 7.00\%) = \$0$$

$$\text{payoff}_{2,L} = \frac{\$1{,}000{,}000}{2} \times (6.50\% - 7.00\%) = -\$2{,}500$$

The possible prices in six months are given by the expected discounted value of the 1-year payoffs under the risk-neutral probabilities, *plus* the 6-month payoffs ($1,250 and –$1,250). Hence, the 6-month values for the top and bottom node are, respectively:

$$V_{1,U} = \frac{(\$2{,}500 \times 0.6) + (\$0 \times 0.4)}{1 + \frac{0.0725}{2}} + \$1{,}250 = \$2{,}697.53$$

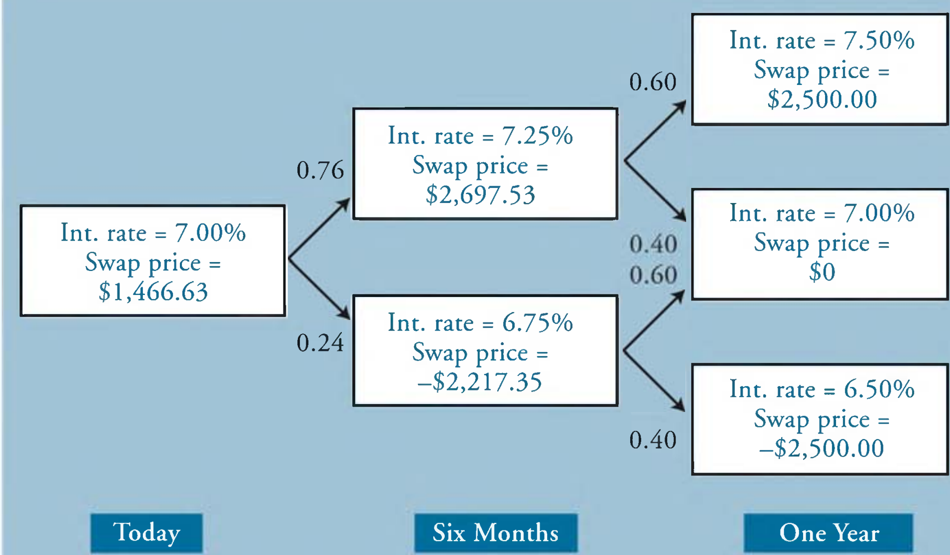
$$V_{1,L} = \frac{(\$0 \times 0.6) + (-\$2{,}500 \times 0.4)}{1 + \frac{0.0675}{2}} - \$1{,}250 = -\$2{,}217.35$$

Today the price is $1,466.63, calculated as follows:

$$V_0 = \frac{(\$2{,}697.53 \times 0.76) + (-\$2{,}217.35 \times 0.24)}{1 + \frac{0.07}{2}} = \$1{,}466.63$$

Figure 6 shows the binomial tree with all values included.

Figure 6: Completed Binomial Tree for CMT Swap



Option-Adjusted Spread

LO 11.6: Define option-adjusted spread (OAS) and apply it to security pricing.

The option-adjusted spread (OAS) is the spread that makes the model value (calculated by the present value of projected cash flows) equal to the current market price. In the previous CMT example, the model price was equal to $1,466.63. Now assume that the market price of the CMT swap was instead $1,464.40, which is $2.23 less than the model price. In this case, the OAS to be added to each discounted risk-neutral rate in the CMT swap binomial tree turns out to be 20 basis points. In six months, the rates to be adjusted are 7.25% in the up node and 6.75% in the down node. Incorporating the OAS into the six-month rates generates the following new swap values:

$$V_{1,U} = \frac{(\$2,500 \times 0.6) + (\$0 \times 0.4)}{1 + \frac{0.0745}{2}} + \$1,250 = \$2,696.13$$

$$V_{1,L} = \frac{(\$0 \times 0.6) + (-\$2,500 \times 0.4)}{1 + \frac{0.0695}{2}} - \$1,250 = -\$2,216.42$$

Notice that the only rates adjusted by the OAS spread are the rates used for discounting values. The OAS does not impact the rates used for estimating cash flows. The final step in this CMT swap valuation is to adjust the interest rate used to discount the price back to today. In this example, the discounted rate of 7% is adjusted by 20 basis points to 7.2%. The updated initial CMT swap value is:

$$V_0 = \frac{(\$2{,}696.13 \times 0.76) + (-\$2{,}216.42 \times 0.24)}{1 + \frac{0.072}{2}} = \$1{,}464.40$$

Now we can see that adding the OAS to the discounted risk-neutral rates in the binomial tree generates a model price ($1,464.40) that is equal to the market price ($1,464.40). In this example, the market price was initially less than the model price. This means that the security was trading *cheap*. If the market price were instead higher than the model price we would say that the security was trading *rich*.

Time Steps

LO 11.9: Evaluate the advantages and disadvantages of reducing the size of the time steps on the pricing of derivatives on fixed income securities.

For the sake of simplicity, the previous example assumed periods of six months. However, in reality, the time between steps should be much smaller. As you can imagine, the smaller the time between steps, the more complicated the tree and calculations become. Using daily time steps will greatly enhance the accuracy of any model but at the expense of additional computational complexity.

Fixed-Income Securities and Black-Scholes-Merton

LO 11.10: Evaluate the appropriateness of the Black-Scholes-Merton model when valuing derivatives on fixed income securities.

The Black-Scholes-Merton model is the most well-known equity option-pricing model. Unfortunately, the model is based on three assumptions that do not apply to fixed-income securities:

1. The model's main shortcoming is that it assumes there is no upper limit to the price of the underlying asset. However, bond prices do have a maximum value. This upper limit occurs when interest rates equal zero so that zero-coupon bonds are priced at par and coupon bonds are priced at the sum of the coupon payments plus par.

2. It assumes the risk-free rate is constant. However, changes in short-term rates do occur, and these changes cause rates along the yield curve and bond prices to change.

3. It assumes bond price volatility is constant. With bonds, however, price volatility decreases as the bond approaches maturity.

Bonds With Embedded Options

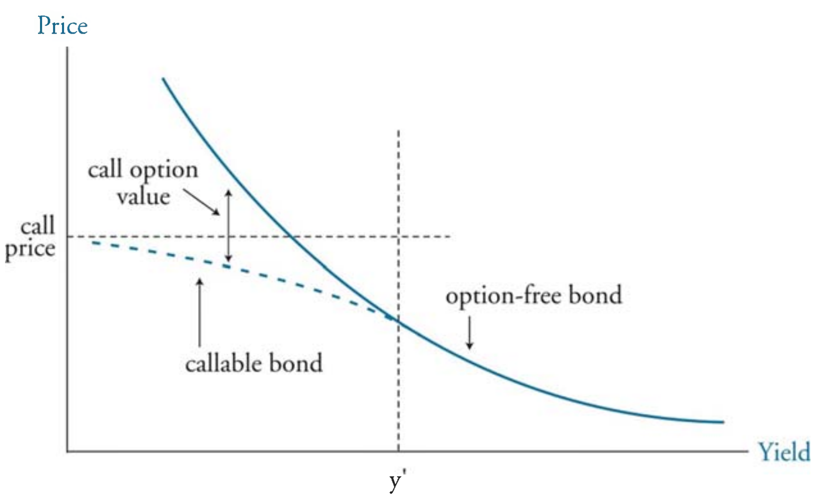
LO 11.11: Describe the impact of embedded options on the value of fixed income securities.

Fixed-income securities are often issued with **embedded options**, such as a call feature. In this case, the price-yield relationship will change, and so will the price volatility characteristics of the issue.

Callable Bonds

A call option gives the issuer the right to buy back the bond at fixed prices at one or more points in the future, prior to the date of maturity. Since the investor takes a short position in the call, the right to purchase rests with the issuer. Such bonds are deemed to be *callable* (note that a call provision on a bond is analytically similar to a prepayment option).

Figure 7: Price-Yield Function of Callable Bond



For an option-free noncallable bond, prices will fall as yields rise, and prices will rise unabated as yields fall—in other words, they'll move in line with yields. That's not the case, however, with **callable bonds**. As you can see in Figure 7, the decline in callable bond yield will reach the point where the rate of increase in the price of the callable bond will start slowing down and eventually level off.

This is known as **negative convexity**. Such behavior is due to the fact that the issuer has the right to retire the bond prior to maturity at some specified call price. The call price, in effect, acts to hold down the price of the bond (as rates fall) and causes the price-yield curve to flatten. The point where the curve starts to flatten is at (or near) a yield level of y'. Note that as long as yields remain above y', a callable bond will behave like any option-free (noncallable) issue and exhibit positive convexity. That's because at high yield levels, there is little chance of the bond being called.

Below y', investors begin to anticipate that the firm may call the bond, in which case investors will receive the call price. Therefore, as yield levels drop, the bond's market value is

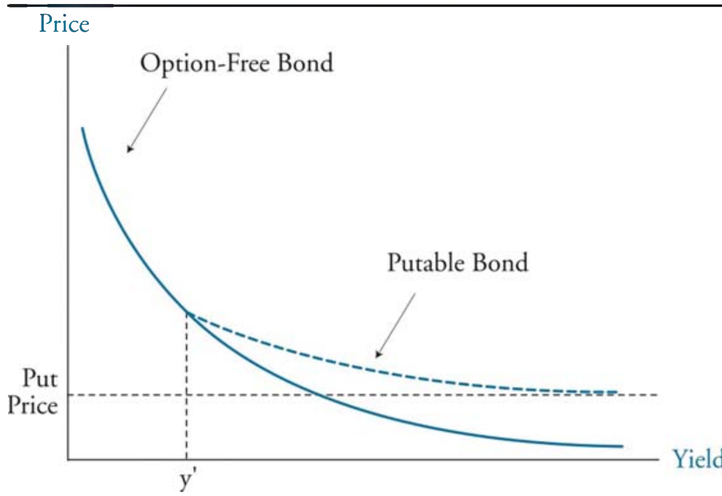
bounded from above by the call price. Thus, callability effectively caps the investor's capital gains as yields fall. Moreover, it exacerbates reinvestment risk since it increases the cash flow that must be reinvested at lower rates (i.e., without the call or prepayment option, the cash flow will only be the coupon; with the option, the cash flow is the coupon plus the call price).

Thus, in Figure 7, as long as yields remain below y', callable bonds will exhibit price compression, or *negative convexity*; however, at yields above y', those same callable bonds will exhibit all the properties of *positive convexity*.

Putable Bonds

The put feature in **putable bonds** is another type of embedded option. The put feature gives the bondholder the right to sell the bond back to the issuer at a set price (i.e., the bondholder can "put" the bond to the issuer). The impact of the put feature on the price-yield relationship is shown in Figure 8.

Figure 8: Price-Yield Function of a Putable Bond



At low yield levels relative to the coupon rate, the price-yield relationship of putable and nonputable bonds is similar. However, as shown in Figure 8, if yields rise above y', the price of the putable bond does not fall as rapidly as the price of the option-free bond. This is because the put price serves as a floor value for the price of the bond.

Key Concepts

LO 11.1

Backward-induction methodology with a binomial model requires discounting of the cash flows that occur at each node in an interest rate tree (bond value plus coupon payment) backward to the root of the tree.

LO 11.2

The values for on-the-run issues generated using an interest rate tree should prohibit arbitrage opportunities.

LO 11.3

Risk-neutral, or no-arbitrage, binomial tree models are used to allow for proper valuation of bonds with embedded options.

LO 11.4

Using the average of present values of the two possible values from the next period may not produce an expected discounted bond value that exactly matches the market price of the bond.

Using 0.5 probabilities for up and down states are the assumed true probabilities of price movements. In order to equate the discounted value using a binomial tree and the market price, we need to use risk-neutral probabilities.

LO 11.5

To value an option on a fixed-income instrument using a binomial tree:

1. Price the bond at each node using projected interest rates.
2. Calculate the intrinsic value of the derivative at each node at maturity.
3. Calculate the expected discounted value of the derivative at each node using the risk-neutral probabilities and working backward through the tree.

Callable bonds can be valued by modifying the cash flows at each node in the interest rate tree to reflect the cash flow prescribed by the embedded call option.

LO 11.6

The option-adjusted spread (OAS) allows a security's model price to equal its market price. It is added to any rate in the interest rate tree that is used for discounting purposes.

LO 11.7

Nonrecombining trees result when the up-down scenario produces a different rate than the down-up scenario.

LO 11.8

A constant maturity Treasury (CMT) swap is an agreement to swap a floating rate for a Treasury rate.

LO 11.9

The precision of a model can be improved by reducing the length of the time steps, but the trade-off is increased complexity.

LO 11.10

The Black-Scholes-Merton model cannot be used for the valuation of fixed-income securities because it makes the following unreasonable assumptions:

- There is no upper price bound.
- The risk-free rate is constant.
- Bond volatility is constant.

LO 11.11

Fixed-income securities are often issued with embedded options. When embedded options are present, the price-yield relationship will change, and so will the price volatility characteristics of the issue.

Concept Checkers

1. A European put option has two years to expiration and a strike price of \$101.00. The underlying is a 7% annual coupon bond with three years to maturity. Assume that the risk-neutral probability of an up move is 0.76 in year 1 and 0.60 in year 2. The current interest rate is 3.00%. At the end of year 1, the rate will either be 5.99% or 4.44%. If the rate in year 1 is 5.99%, it will either rise to 8.56% or rise to 6.34% in year 2. If the rate in one year is 4.44%, it will either rise to 6.34% or rise to 4.70%. The value of the put option today is closest to:
   A. \$1.17.
   B. \$1.30.
   C. \$1.49.
   D. \$1.98.

2. The Black-Scholes-Merton option pricing model is not appropriate for valuing options on corporate bonds because corporate bonds:
   A. have credit risk.
   B. have an upper price bound.
   C. have constant price volatility.
   D. are not priced by arbitrage.

3. Which of the following regarding the use of small time steps in the binomial model is true?
   A. Less realistic model.
   B. More accurate model.
   C. Less complicated computations.
   D. Less computational expense.

4. Which of the following statements about callable bonds compared to noncallable bonds is false?
   A. They have less price volatility.
   B. They have negative convexity.
   C. Capital gains are capped as yields rise.
   D. At low yields, reinvestment rate risk rises.

5. Which of the following statements concerning the calculation of value at a node in a fixed income binomial interest rate tree is most accurate? The value at each node is the:
   A. present value of the two possible values from the next period.
   B. average of the present values of the two possible values from the next period.
   C. sum of the present values of the two possible values from the next period.
   D. average of the future values of the two possible values from the next period.

Concept Checker Answers

1. **A** This is the same underlying bond and interest rate tree as in the call option example from this topic. However, here we are valuing a put option.

   The option value in the upper node at the end of year 1 is computed as:

   $$\frac{(\$2.44 \times 0.6) + (\$0.38 \times 0.4)}{1.0599} = \$1.52$$

   The option value in the lower node at the end of year 1 is computed as:

   $$\frac{(\$0.38 \times 0.6) + (\$0.00 \times 0.4)}{1.0444} = \$0.22$$

   The option value today is computed as:

   $$\frac{(\$1.52 \times 0.76) + (\$0.22 \times 0.24)}{1.0300} = \$1.17$$

2. **B** The Black-Scholes-Merton model cannot be used for the valuation of fixed-income securities because it makes the following assumptions, which are not reasonable for valuing fixed-income securities:
   - There is no upper price bound.
   - The risk-free rate is constant.
   - Bond volatility is constant.

3. **B** The use of small time steps in the binomial model yields a more realistic model, a more accurate model, more complicated computations, and more computational expense.

4. **C** Callable bonds have the following characteristics:
   - *Less* price volatility.
   - Negative convexity.
   - Capital gains are capped as yields *fall*.
   - Exhibit increased reinvestment rate risk when yields fall.

5. **B** The value at any given node in a binomial tree is the average present value of the cash flows at the two possible states immediately to the right of the given node, discounted at the 1-period rate at the node under examination.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

The Evolution of Short Rates and the Shape of the Term Structure

Topic 12

Exam Focus

This topic discusses how the decision tree framework is used to estimate the price and returns of zero-coupon bonds. This decision tree framework illustrates how interest rate expectations determine the shape of the yield curve. For the exam, candidates should understand how current spot rates and forward rates are determined by the expectations, volatility, and risk premiums of short-term rates. Furthermore, the use of Jensen's inequality should be understood, and candidates should be prepared to use this formula to demonstrate how maturity and volatility increase the convexity of zero-coupon bonds.

Interest Rate Expectations

LO 12.1: Explain the role of interest rate expectations in determining the shape of the term structure.

Expectations of future interest rates are based on uncertainty. For example, an investor may expect that interest rates over the next period will be 8%. However, this investor may realize there is also a high probability that interest rates could be 7% or 9% over the next period.

Expectations play an important role in determining the shape of the yield curve and can be illustrated by examining yield curves that are flat, upward-sloping, and downward-sloping. In the following yield curve examples assume future interest rates are known and that there is no uncertainty in rates.

Flat Yield Curve

Suppose the 1-year interest rate is 8% and future 1-year forward rates are 8% for the next two years. Given these interest rate expectations, the present values of 1-, 2-, and 3-year zero-coupon bonds with $1 face values assuming annual compounding are calculated as follows:

$$\text{price of 1-year zero-coupon bond} = \frac{\$1}{1.08} = \$0.92593$$

$$\text{price of 2-year zero-coupon bond} = \frac{\$1}{(1.08)(1.08)} = \$0.85734$$

$$\text{price of 3-year zero-coupon bond} = \frac{\$1}{(1.08)(1.08)(1.08)} = \$0.79383$$

In this example, investors expect the 1-year spot rates for the next three years to be 8%. Thus, the yield curve is flat and investors are willing to lock in interest rates for two or three years at 8%.

Upward-Sloping Yield Curve

Now suppose the 1-year interest rate will remain at 8%, but investors expect the 1-year rate in one year to be 10% and the 1-year rate in two years to be 12%. The 2-year spot rate, $\hat{r}(2)$, must satisfy the following equation:

$$\text{price of 2-year zero-coupon bond} = \frac{\$1}{(1.08)(1.10)} = \frac{\$1}{(1+\hat{r}(2))^2}$$

Cross multiplying, taking the square root of each side, and subtracting one from each side results in the following equation:

$$\hat{r}(2) = \sqrt[2]{(1.08)(1.10)} - 1 = \sqrt{1.188} - 1$$

Solving this equation reveals that the 2-year spot rate is 8.995%.

The 3-year spot rate can be solved in a similar fashion:

$$\text{price of 3-year zero-coupon bond} = \frac{\$1}{(1.08)(1.10)(1.12)} = \frac{\$1}{(1+\hat{r}(3))^3}$$

Thus, the 3-year spot rate is computed as follows:

$$\hat{r}(3) = \sqrt[3]{(1.08)(1.10)(1.12)} - 1 = (1.33056)^{1/3} - 1 = 9.988\%$$

If expected 1-year spot rates for the next three years are 8%, 10%, and 12%, then this results in an upward-sloping yield curve of 1-, 2-, and 3-year spot rates of 8%, 8.995%, and 9.988%, respectively. Thus, the expected future spot rates will determine the upward-sloping shape of the yield curve.

Downward-Sloping Yield Curve

Now suppose the 1-year interest rate will remain at 8%, but investors expect the 1-year rate in one year to be 6% and the 1-year rate in two years to be 4%. The 2-year and 3-year spot rates are computed as follows:

$$\hat{r}(2) = \sqrt[2]{(1.08)(1.06)} - 1 = 6.995\%$$

$$\hat{r}(3) = \sqrt[3]{(1.08)(1.06)(1.04)} - 1 = 5.987\%$$

These calculations imply a downward-sloping yield curve for 1-, 2-, and 3-year spot rates of 8%, 6.995%, and 5.987%, respectively.

These three examples illustrate that expectations of future interest rates can describe the shape and level of the term structure for short-term horizons. If expected 1-year spot rates for the next three years are r_1, r_2, and r_3, then the 2-year and 3-year spot rates are computed as:

$$\hat{r}(2) = \sqrt[2]{(1 + r_1)(1 + r_2)} - 1$$

$$\hat{r}(3) = \sqrt[3]{(1 + r_1)(1 + r_2)(1 + r_3)} - 1$$

In the short run, expected future spot rates determine the shape of the yield curve. In the long run, however, it is less likely that investors have confidence in 1-year spot rates several years from now (e.g., 30 years). Thus, expectations are unable to describe the shape of the term structure for long-term horizons. However, it is reasonable to assume that real rates and inflation rates are relatively constant over the long run. For example, a short-term rate of 5% may imply a long-run real rate of interest of 3% and a long-run inflation rate of 2%. Thus, interest rate expectations can describe the level of interest rates for long-term horizons.

Interest Rate Volatility

LO 12.2: Apply a risk-neutral interest rate tree to assess the effect of volatility on the shape of the term structure.

Suppose an investor is risk-neutral and has uncertainty regarding expectations of future interest rates. The decision tree in Figure 1 represents expectations that are used to determine the 1-year rate. If there is a 50% probability that the 1-year rate in one year will be 10% and a 50% probability that the 1-year rate in one year will 6%, the expected 1-year rate in one year can be calculated as:

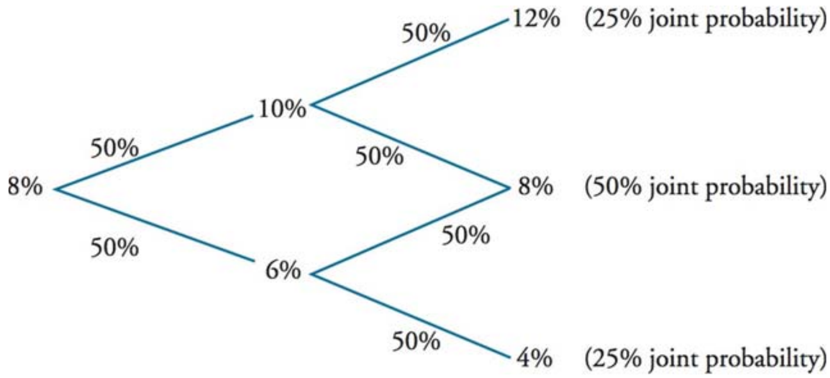
$$(0.5 \times 10\%) + (0.5 \times 6\%) = 8\%$$

Using the expected rates in Figure 1, the price of a 1-year zero-coupon bond is 0.92593% of par [calculated as $1 / 1.08 with a $1 par value].

The last column of the decision tree in Figure 1 illustrates the joint probabilities of 12%, 8%, or 4% 1-year rates in two years. The 12% rate in the upper node occurs 25% of the time (50% × 50%), the 8% rate in the middle node occurs 50% of the time (50% × 50% + 50% × 50%), and the 4% rate in the bottom node occurs 25% of the time (50% × 50%). Thus, the expected 1-year rate in two years is calculated as:

$$(0.25 \times 12\%) + (0.50 \times 8\%) + (0.25 \times 4\%) = 8\%$$

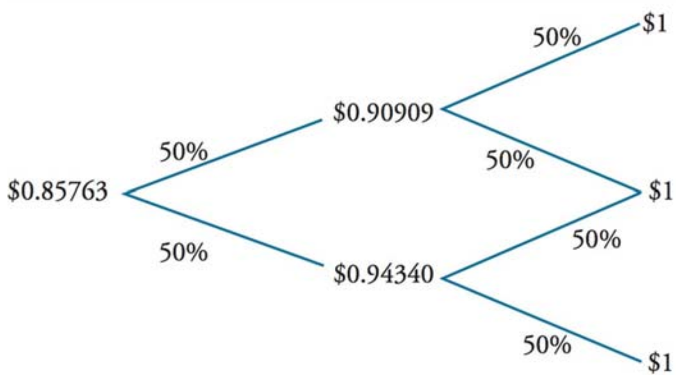
Figure 1: Decision Tree Illustrating Expected 1-Year Rates for Two Years



Assuming risk-neutrality, the decision tree in Figure 2 illustrates the calculation of the price of a 2-year zero-coupon bond with a face value of \$1 using the expected rates given in the Figure 1 decision tree. The expected price in one year for the upper node is \$0.90909, calculated as \$1 / 1.10. The expected price in one year for the lower node is \$0.94340, calculated as \$1 / 1.06. Thus, the current price of the 2-year zero-coupon bond is calculated as:

$$[0.5 \times (\$0.90909 / 1.08)] + [0.5 \times (\$0.94340 / 1.08)] = \$0.85763$$

Figure 2: Risk-Neutral Decision Tree for a 2-Year Zero-Coupon Bond



With the present value of the 2-year zero-coupon bond, we can computed the implied 2-year spot rate by solving for $\hat{r}(2)$ as follows:

$$\$0.85763 = \frac{\$1}{(1+\hat{r}(2))^2}$$

$$\hat{r}(2) = \sqrt{\frac{1}{0.85763}} - 1 = 0.079816 \text{ or } 7.9816\%$$

Alternatively, this can also be computed using a financial calculator as follows:

PV = –0.85763; FV = 1; PMT = 0; N = 2; CPT → I/Y = 7.9816%

This example illustrates that when there is uncertainty regarding expected rates, the volatility of expected rates causes the future spot rates to be lower. With the implied rate, we can compute the value of convexity for the 2-year zero-coupon bond as: 8% – 7.9816% = 0.0184% or 1.84 basis points.

Convexity Effect

LO 12.3: Estimate the convexity effect using Jensen's inequality.

LO 12.4: Evaluate the impact of changes in maturity, yield, and volatility on the convexity of a security.

The convexity effect can be measured by applying a special case of Jensen's inequality as follows:

$$E\left[\frac{1}{(1+r)}\right] > \frac{1}{E[1+r]}$$

Example: Applying Jensen's inequality

Assume that next year there is a 50% probability that 1-year spot rates will be 10% and a 50% probability that 1-year spot rates will be 6%. **Demonstrate** Jensen's inequality for a 2-year zero-coupon bond with a face value of $1 assuming the previous interest rate expectations shown in Figure 1.

Answer:

The left-hand side of Jensen's inequality is the expected price in one year using the 1-year spot rates of 10% and 6%.

$$E\left[\frac{\$1}{(1+r)}\right] = 0.5 \times \frac{\$1}{(1.10)} + 0.5 \times \frac{\$1}{(1.06)} = \$0.92624$$

The expected price in one year using an expected rate of 8% computes the right-hand side of the inequality as:

$$\frac{\$1}{0.5 \times 1.10 + 0.5 \times 1.06} = \frac{\$1}{1.08} = 0.92593$$

Thus, the left-hand side is greater than the right-hand side, $0.92624 > $0.92593.

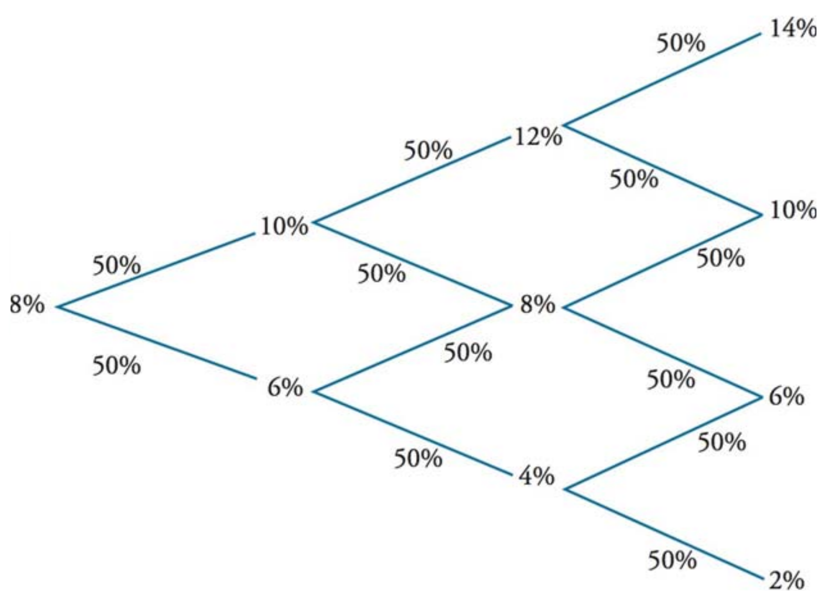
If the current 1-year rate is 8%, then the price of a 2-year zero-coupon bond is found by simply dividing each side of the equation by 1.08. In other words, discount the expected 1-year zero-coupon bond price for one more year at 8% to find the 2-year price. The price of the 2-year zero-coupon bond on the left-hand side of Jensen's inequality equals \$0.85763 (calculated as \$0.92624 / 1.08). The right-hand side is calculated as the price of a 2-year zero-coupon bond discounted for two years at the expected rate of 8%, which equals \$0.85734 (calculated as \$1 / 1.08^2).

The left-hand side is again greater than the right-hand side, \$0.85763 > \$0.85734.

This demonstrates that the price of the 2-year zero-coupon bond is greater than the price obtained by discounting the \$1 face amount by 8% over the first period and by 8% over the second period. Therefore, we know that since the 2-year zero-coupon price is higher than the price achieved through discounting, its implied rate must be lower than 8%.

Extending the above example out for one more year illustrates that convexity increases with maturity. Suppose an investor expects the spot rates to be 14%, 10%, 6%, or 2% in three years. Assuming each expected return has an equal probability of occurring results in the decision tree shown in Figure 3.

Figure 3: Risk-Neutral Decision Tree Illustrating Expected 1-Year Rates for Three Years



The decision tree in Figure 4 uses the expected spot rates from the decision tree in Figure 3 to calculate the price of a 3-year zero-coupon bond.

The price of a 1-year zero-coupon bond in two years with a face value of \$1 for the upper node is \$0.89286 (calculated as \$1 / 1.12). The price of a 1-year zero coupon bond in two years for the middle node is \$0.92593 (calculated as \$1 / 1.08). The price of a 1-year zero coupon bond in two years for the bottom node is \$0.96154 (calculated as \$1 / 1.04).

The price of a 2-year zero-coupon bond in one year using the upper node expected spot rates is calculated as:

[0.5 × ($0.89286 / 1.10)] + [0.5 × ($0.92593 / 1.10)] = $0.82672

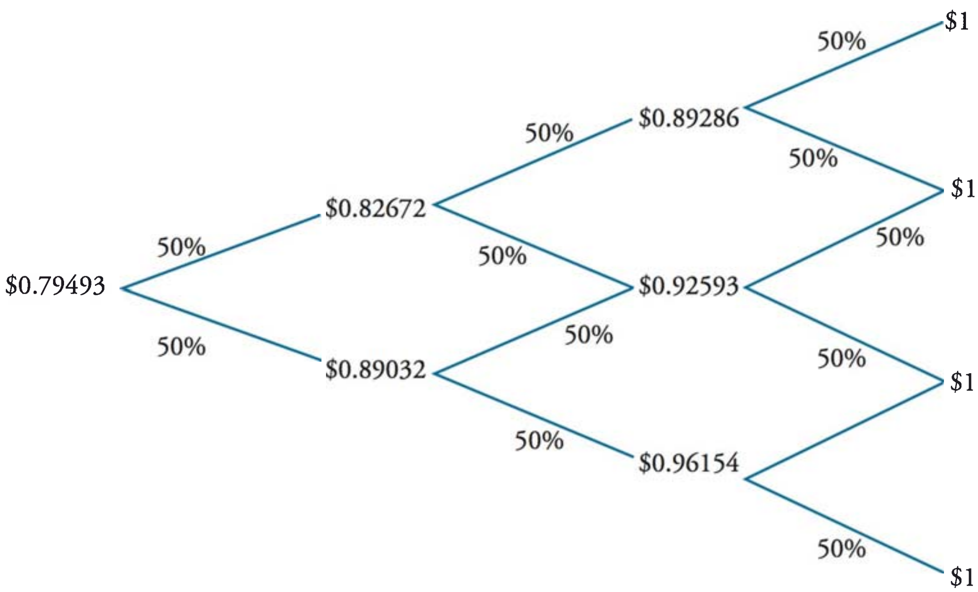
The price of a 2-year zero-coupon bond in one year using the bottom node expected spot rates is calculated as:

[0.5 × ($0.92593 / 1.06)] + [0.5 × ($0.96154 / 1.06)] = $0.89032

Lastly, the price of a 3-year zero-coupon bond today is calculated as:

[0.5 × ($0.82672 / 1.08)] + [0.5 × ($0.89032 / 1.08)] = $0.79493

Figure 4: Risk-Neutral Decision Tree for a 3-Year Zero-Coupon Bond



To measure the convexity effect, the implied 3-year spot rate is calculated by solving for $\hat{r}(3)$ in the following equation:

$$0.79493 = \frac{1}{(1 + r(3))^3}$$

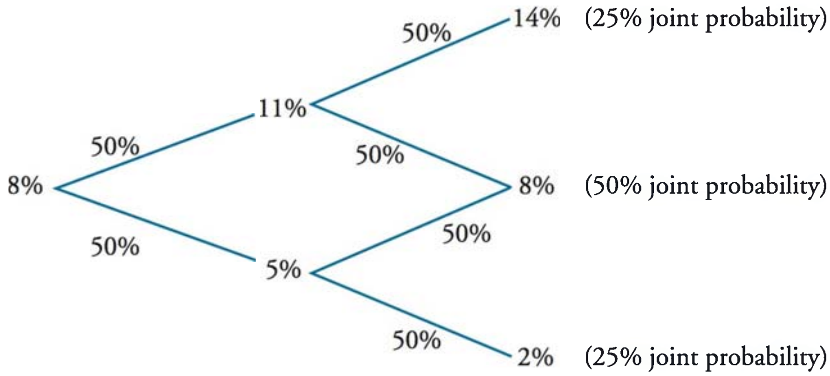
$$\hat{r}(3) = \sqrt[3]{\frac{1}{0.79493}} - 1 = 0.0795 \text{ or } 7.95\%$$

Notice that convexity lowers bond yields and that this reduction in yields is equal to the value of convexity. For the 3-year zero-coupon bond, the value of convexity is 8% – 7.95% = 0.05% or 5 basis points. Recall that the value of convexity for the 2-year zero-coupon bond was only 1.84 basis points. Therefore, *all else held equal, the value of convexity*

increases with maturity. In other words, as the maturity of a bond increases, the price-yield relationship becomes more convex.

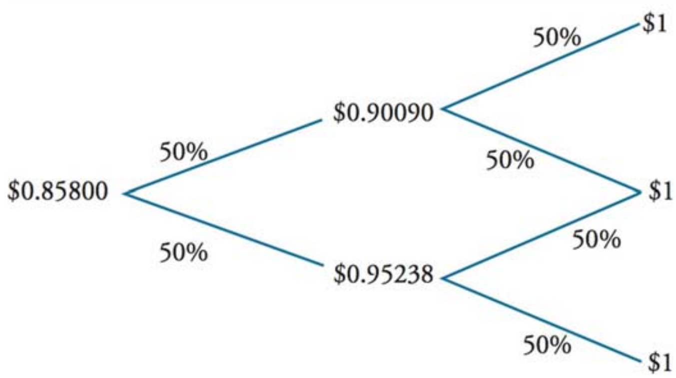
This convexity occurs due to volatility. Thus, we can also say that the value of convexity increases with volatility. The following decision trees in Figures 5 and 6 illustrate the impact of increasing the volatility of interest rates. In this example, the 1-year spot rate in one year in Figure 5 ranges from 2% to 14% instead of 4% to 12% as was shown in Figure 1.

Figure 5: Risk-Neutral Decision Tree Illustrating Volatility Effect on Convexity



Using the same methodology as before, the price of a 2-year zero-coupon bond with the listed expected interest rates in Figure 5 is $0.858.

Figure 6: Price of a 2-Year Zero-Coupon Bond with Increased Volatility



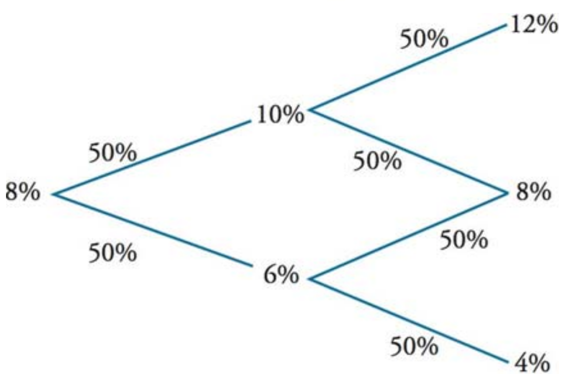
This price results in a 2-year implied spot rate of 7.958%. Thus, the value of convexity is 8% – 7.958% = 0.042% or 4.2 basis points. This is higher than the previous 2-year example where the value of convexity was 1.84 basis points when expected spot rates ranged from 4% to 12%, instead of 2% to 14%. Therefore, *the value of convexity increases with both volatility and time to maturity*.

Risk Premium

LO 12.5: Calculate the price and return of a zero coupon bond incorporating a risk premium.

Suppose an investor expects 1-year rates to resemble those in Figure 7. In this example, there is volatility of 400 basis point of rates per year where 1-year rates in one year range from 4% to 12% in the second year.

Figure 7: Decision Tree Illustrating Expected 1-Year Rates for Two Years



Next year, the 1-year return will be either 10% or 6%. A risk-neutral investor calculates the price of a 2-year zero-coupon bond with a face value of $1 as follows:

$$\frac{\left[\frac{\$1}{1.10}+\frac{\$1}{1.06}\right]\times 0.5}{1.08}=\frac{[\$0.90909+\$0.94340]\times 0.5}{1.08}=\$0.85763$$

In this example, the price of $0.85763 implies a 1-year expected return of 8%. However, this is only the average return. The actual return will be either 6% or 10%. Risk-averse investors would require a higher rate of return for this investment than an investment that has a certain 8% return with no variability. Thus, risk-averse investors require a risk premium for bearing this interest rate risk, and demand a return greater than 8% for buying a 2-year zero-coupon bond and holding it for the next year.

Example: Incorporating a risk premium

Calculate the price and return for the zero-coupon bond using the expected returns in Figure 7 and assuming a risk premium of 30 basis points for each year of interest rate risk.

Answer:

The price of a 2-year zero-coupon bond with a 30 basis point risk premium included is calculated as:

$$\frac{\left[\frac{\$1}{1.103}+\frac{\$1}{1.063}\right]\times 0.5}{1.08}=\frac{[\$0.90662+\$0.94073]\times 0.5}{1.08}=\$0.85525$$

Notice that this price is less than the \$0.85763 price calculated previously for the risk-neutral investor. Next year, the price of the 2-year zero-coupon bond will either be \$0.90909 or \$0.94340, depending on whether the 1-year rate is either 10% or 6%, respectively. Thus, the expected return for the next year of the 2-year zero-coupon bond is 8.3%, calculated as follows:

$$\frac{(\$0.90909+\$0.94340)\times 0.5-\$0.85525}{\$0.85525}=0.083$$

Therefore, risk-averse investors require a 30 basis point premium or 8.3% return to compensate for one year of interest rate risk. For a 3-year zero-coupon bond, risk-averse investors will require a 60 basis point premium or 8.6% return given two years of interest rate risk.

Professor's Note: In the previous example, it is assumed that rates can change only once a year, so in the first year there is no uncertainty of interest rates. There is only uncertainty in what the 1-year rate will be one and two years from today.

Key Concepts

LO 12.1

If expected 1-year spot rates for the next three years are r_1, r_2, and r_3, then the 2-year spot rate, $\hat{r}(2)$, is computed as $\hat{r}(2) = \sqrt[2]{(1+r_1)(1+r_2)} - 1$, and the 3-year spot rate, $\hat{r}(3)$, is computed as $\hat{r}(3) = \sqrt[3]{(1+r_1)(1+r_2)(1+r_3)} - 1$.

LO 12.2

The volatility of expected rates creates convexity, which lowers future spot rates.

LO 12.3

The convexity effect can be measured by using Jensen's inequality: $E\left[\frac{1}{(1+r)}\right] > \frac{1}{E[1+r]}$.

LO 12.4

Convexity lowers bond yields due to volatility. This reduction in yields is equal to the value of convexity. Thus, we can say that the value of convexity increases with volatility. The value of convexity will also increase with maturity, because the price-yield relationship will become more convex over time.

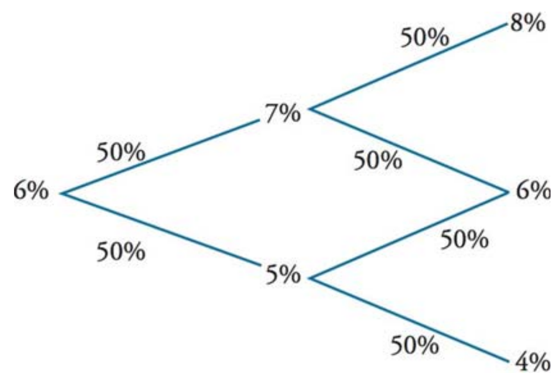
LO 12.5

Risk-averse investors will price bonds with a risk premium to compensate them for taking on interest rate risk.

Concept Checkers

1. An investor expects the current 1-year rate for a zero-coupon bond to remain at 6%, the 1-year rate next year to be 8%, and the 1-year rate in two years to be 10%. What is the 3-year spot rate for a zero-coupon bond with a face value of $1, assuming all investors have the same expectations of future 1-year rates for zero-coupon bonds?
A. 7.888%.
B. 7.988%.
C. 8.000%.
D. 8.088%.

2. Suppose investors have interest rate expectations as illustrated in the decision tree below where the 1-year rate is expected to be 8%, 6%, or 4% in the second year and either 7% or 5% in the first year for a zero-coupon bond.



If investors are risk-neutral, what is the price of a $1 face value 2-year zero-coupon bond today?
A. $0.88113.
B. $0.88634.
C. $0.89007.
D. $0.89032.

3. If investors are risk-neutral and the price of a 2-year zero-coupon bond is $0.88035 today, what is the implied 2-year spot rate?
A. 4.339%.
B. 5.230%.
C. 5.827%.
D. 6.579%.

4. What is the impact on the bond price-yield curve if, all other factors held constant, the maturity of a zero-coupon bond increases? The pricing curve becomes:
A. less concave.
B. more concave.
C. less convex.
D. more convex.

5. Suppose an investor expects that the 1-year rate will remain at 6% for the first year for a 2-year zero-coupon bond. The investor also projects a 50% probability that the 1-year spot rate will be 8% in one year and a 50% probability that the 1-year spot rate will be 4% in one year. Which of the following inequalities most accurately reflects the convexity effect for this 2-year bond using Jensen's inequality formula?
   A. \$0.89031 > \$0.89000.
   B. \$0.89000 > \$0.80000.
   C. \$0.94340 > \$0.89031.
   D. \$0.94373 > \$0.94340.

Concept Checker Answers

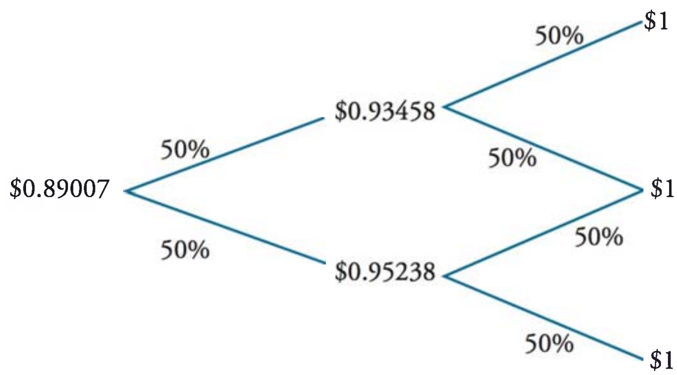
1. **B** The 3-year spot rate can be solved for using the following equation:

$$\frac{\$1}{(1.06)(1.08)(1.10)} = \frac{\$1}{(1+\hat{r}(3))^3}$$

Solving for $\hat{r}(3) = \sqrt[3]{(1.06)(1.08)(1.10)} - 1 = 7.988\%$

2. **C** Assuming investors are risk-neutral, the following decision tree illustrates the calculation of the price of a 2-year zero-coupon bond using the expected rates given. The expected price in one year for the upper node is \$0.93458, calculated as \$1 / 1.07. The expected price in one year for the lower node is \$0.95238, calculated as \$1 / 1.05. Thus, the current price is \$0.89007, calculated as:

[0.5 × (\$0.93458 / 1.06)] + [0.5 × (\$0.95238 / 1.06)] = \$0.89007



3. **D** The implied 2-year spot rate is calculated by solving for $\hat{r}(2)$ in the following equation:

$$\$0.88035 = \frac{\$1}{(1+\hat{r}(2))^2}$$

$$\hat{r}(2) = \sqrt{\frac{1}{0.88035}} - 1 = 0.06579 \text{ or } 6.579\%$$

Alternatively, this can also be computed using a financial calculator as follows:

PV = –0.88035; FV = 1; PMT = 0; N = 2; CPT → I/Y = 6.579%.

4. **D** As the maturity of a bond increases, the price-yield relationship becomes more convex.

5. **A** The left-hand side of Jensen's inequality is the expected price in one year using the 1-year spot rates of 8% and 4%.

$$E\left[\frac{\$1}{(1+r)}\right] = 0.5 \times \frac{\$1}{(1.08)} + 0.5 \times \frac{\$1}{(1.04)} = 0.5 \times 0.92593 + 0.5 \times \$0.96154 = \$0.94373$$

The expected price in one year using an expected rate of 6% computes the right-hand side of the inequality as:

$$\frac{\$1}{0.5 \times 1.08 + 0.5 \times 1.04} = \frac{\$1}{1.06} = 0.94340$$

Next, divide each side of the equation by 1.06 to discount the expected 1-year zero-coupon bond price for one more year at 6%. The price of the 2-year zero-coupon bond equals \$0.89031 (calculated as \$0.94373 / 1.06), which is greater than \$0.89000 (the price of a 2-year zero-coupon bond discounted for two years at the expected rate of 6%). Thus, Jensen's inequality reveals that \$0.89031 > \$0.89000.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

The Art of Term Structure Models: Drift

Topic 13

Exam Focus

This topic introduces different term structure models for estimating short-term interest rates. Specifically, we will discuss models that have no drift (Model 1), constant drift (Model 2), time-deterministic drift (Ho-Lee), and mean-reverting drift (Vasicek). For the exam, understand the differences between these short rate models, and know how to construct a two-period interest rate tree using model predictions. Also, know how the limitations of each model impact model effectiveness. For the Vasicek model, understand how to convert a nonrecombining tree into a combining tree.

LO 13.1: Construct and describe the effectiveness of a short term interest rate tree assuming normally distributed rates, both with and without drift.

Term Structure Model with No Drift (Model I)

This topic begins with the simplest model for predicting the evolution of short rates (Model 1), which is used in cases where there is no drift and interest rates are normally distributed. The continuously compounded instantaneous rate, denoted r_t, will change (over time) according to the following relationship:

$$dr = \sigma dw$$

where:
dr = change in interest rates over small time interval, dt
dt = small time interval (measured in years) (e.g., one month = 1/12)
σ = annual basis-point volatility of rate changes
dw = normally distributed random variable with mean 0 and standard deviation $\sqrt{dt}$

Given this definition, we can build an interest rate tree using a binomial model. The probability of up and down movements will be the same from period to period (50% up and 50% down) and the tree will be recombining. Since the tree is recombining, the up-down path ends up at the same place as the down-up path in the second time period.

For example, consider the evolution of interest rates on a monthly basis. Assume the current short-term interest rate is 6% and annual volatility is 120bps. Using the above notation, r_0 = 6%, σ = 1.20%, and dt = 1/12. Therefore, *dw* has a mean of 0 and standard deviation of $\sqrt{1/12}$ = 0.2887.

After one month passes, assume the random variable *dw* takes on a value of 0.2 (drawn from a normal distribution with mean = 0 and standard deviation = 0.2887). Therefore, the

change in interest rates over one month is calculated as: dr = 1.20% × 0.2 = 0.24% = 24 basis points. Since the initial rate was 6% and interest rates "changed" by 0.24%, the new spot rate in one month will be: 6% + 0.24% = 6.24%.

LO 13.2: Calculate the short-term rate change and standard deviation of the rate change using a model with normally distributed rates and no drift.

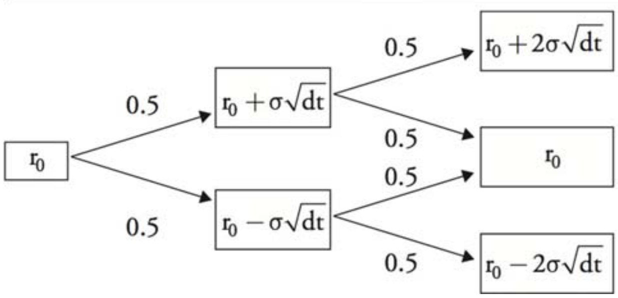
In Model 1, since the expected value of *dw* is zero [i.e., E(dw) = 0], the drift will be zero. Also, since the standard deviation of dw = $\sqrt{dt}$, the volatility of the rate change = $\sigma\sqrt{dt}$. This expression is also referred to as the standard deviation of the rate.

In the preceding example, the standard deviation of the rate is calculated as:

$$1.2\% \times \sqrt{1/12} = 0.346\% = 34.6 \text{ basis points}$$

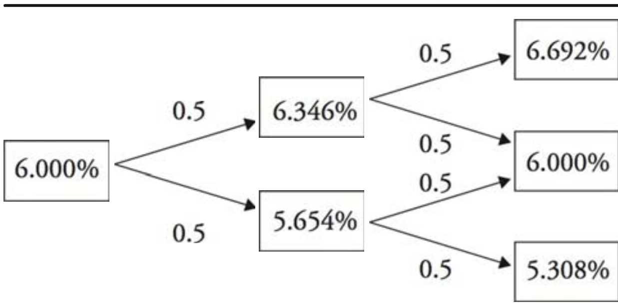
Returning to our previous discussion, we are now ready to construct an interest rate tree using Model 1. A generic interest rate tree over two periods is presented in Figure 1. Note that this tree is recombining and the ending rate at time 2 for the middle node is the same as the initial rate, r_0. Hence, the model has no drift.

Figure 1: Interest Rate Tree with No Drift



The interest rate tree using the previous numerical example is shown in Figure 2. One period from now, the observed interest rate will either increase with 50% probability to: 6% + 0.346% = 6.346% or decrease with 50% probability to: 6% – 0.346% = 5.654%. Extending to two periods completes the tree with upper node: 6% + 2(0.346%) = 6.692%, middle node: 6% (unchanged), and lower node: 6% – 2(0.346%) = 5.308%.

Figure 2: Numerical Example of Interest Rate Tree with No Drift



LO 13.3: Describe methods for addressing the possibility of negative short-term rates in term structure models.

Note that the terminal nodes in the two-period model generate three possible ending rates: $r_0 + 2\sigma\sqrt{dt}$, r_0, and $r_0 - 2\sigma\sqrt{dt}$. This discrete, finite set of outcomes does not technically represent a normal distribution. However, our knowledge of probability distributions tells us that as the number of steps increases, the terminal distribution at the nodes will approach a continuous normal distribution.

One obvious drawback to Model 1 is that there is always a positive probability that interest rates could become negative. On the surface, negative interest rates do not make much economic sense (i.e., lending \$100 and receiving less than \$100 back in the future). However, you could plausibly rationalize a small negative interest rate if the safety and/or inconvenience of holding cash were sufficiently high.

The negative interest rate problem will be exacerbated as the investment horizon gets longer, since it is more likely that forecasted interest rates will drop below zero. As an illustration, assume a ten-year horizon and a standard deviation of terminal interest rates of $1.2\% \times \sqrt{10} = 3.79\%$. It is clear that negative interest rates will be well within a two standard deviation confidence interval when centered around a current rate of 6%. Also note that the problem of negative interest rates is greater when the current level of interest rates is low (e.g., 4% instead of the original 6%).

There are two reasonable solutions for negative interest rates. First, the model could use distributions that are always non-negative, such as lognormal or chi-squared distributions. In this way, the interest rate can never be negative, but this action may introduce other non-desirable characteristics such as skewness or inappropriate volatilities. Second, the interest rate tree can "force" negative interest rates to take a value of zero. In this way, the original interest rate tree is adjusted to constrain the distribution from being below zero. This method may be preferred over the first method because it forces a change in the original distribution only in a very low interest rate environment whereas changing the entire distribution will impact a much wider range of rates.

As a final note, it is ultimately up to the user to decide on the appropriateness of the model. For example, if the purpose of the term structure model is to price coupon-paying bonds, then the valuation is closely tied to the average interest rate over the life of the bond and the possible effect of negative interest rates (small probability of occurring or staying negative for long) is less important. On the other hand, option valuation models that have asymmetric payoffs will be more affected by the negative interest rate problem.

Model 1 Effectiveness

Given the no-drift assumption of Model 1, we can draw several conclusions regarding the effectiveness of this model for predicting the shape of the term structure:

- The no-drift assumption does not give enough flexibility to accurately model basic term structure shapes. The result is a downward-sloping predicted term structure due to a larger convexity effect. Recall that the convexity effect is the difference between the model par yield using its assumed volatility and the par yield in the structural model with assumed zero volatility.

- Model 1 predicts a flat term structure of volatility, whereas the observed volatility term structure is hump-shaped, rising and then falling.
- Model 1 only has one factor, the short-term rate. Other models that incorporate additional factors (e.g., drift, time-dependent volatility) form a richer set of predictions.
- Model 1 implies that any change in the short-term rate would lead to a parallel shift in the yield curve, again, a finding incongruous with observed (non-parallel) yield curve shifts.

Term Structure Model with Drift (Model 2)

Casual term structure observation typically reveals an upward-sloping yield curve, which is at odds with Model 1, which does not incorporate drift. A natural extension to Model 1 is to add a positive drift term that can be economically interpreted as a positive risk premium associated with longer time horizons. We can augment Model 1 with a constant drift term, which yields Model 2:

$$dr = \lambda dt + \sigma dw$$

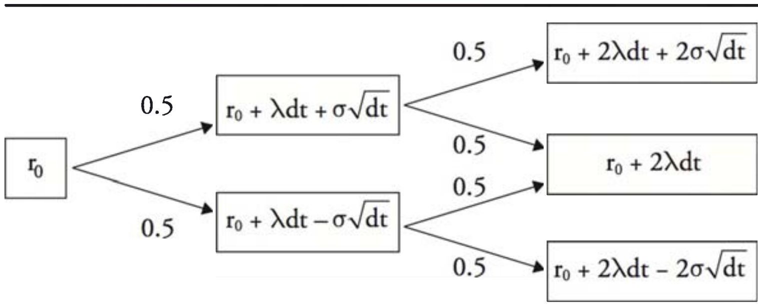
Let's continue with a new example assuming a current short-term interest rate, r_0, of 5%, drift, λ, of 0.24%, and standard deviation, σ, of 1.50%. As before, the *dw* realization drawn from a normal distribution (with mean = 0 and standard deviation = 0.2887) is 0.2. Thus, the change in the short-term rate in one month is calculated as:

$$dr = 0.24\% \times (1/12) + 1.5\% \times 0.2 = 0.32\%$$

Hence, the new rate, r_1, is computed as: 5% + 0.32% = 5.32%. The monthly drift is 0.24% × 1/12 = 0.02% and the standard deviation of the rate is $1.5\% \times \sqrt{1/12} = 0.43\%$ (i.e., 43 basis points per month). The 2bps drift per month (0.02%) represents any combination of expected changes in the short-term rate (i.e., true drift) and a risk premium. For example, the 2bps observed drift could result from a 1.5bp change in rates coupled with a 0.5bp risk premium.

The interest rate tree for Model 2 will look very similar to Model 1, but the drift term, λdt, will increase by λdt in the next period, $2\lambda dt$ in the second period, and so on. This is visually represented in Figure 3. Note that the tree recombines at time 2, but the value at time 2, $r_0 + 2\lambda dt$, is greater than the original rate, r_0, due to the positive drift.

Figure 3: Interest Rate Tree with Constant Drift



Model 2 Effectiveness

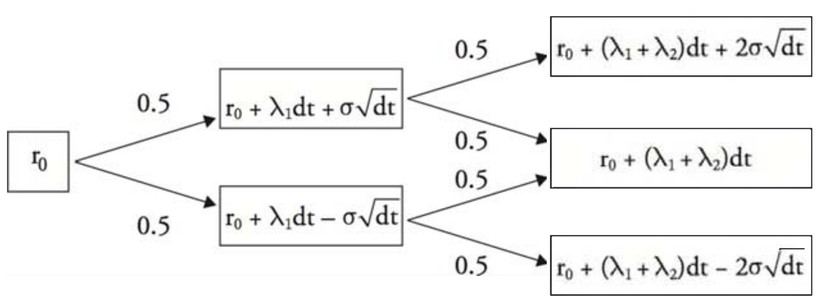
As you would expect, Model 2 is more effective than Model 1. Intuitively, the drift term can accommodate the typically observed upward-sloping nature of the term structure. In practice, a researcher is likely to choose r_0 and λ based on the calibration of observed rates. Hence, the term structure will fit better. The downside of this approach is that the estimated value of drift could be relatively high, especially if considered as a risk premium only. On the other hand, if the drift is viewed as a combination of the risk premium and the expected rate change, the model suggests that the expected rates in year 10 will be higher than year 9, for example. This view is more appropriate in the short run, since it is more difficult to justify increases in expected rates in the long run.

Ho-Lee Model

LO 13.4: Construct a short-term rate tree under the Ho-Lee Model with time-dependent drift.

The Ho-Lee model further generalizes the drift to incorporate time-dependency. That is, the drift in time 1 may be different than the drift in time 2; additionally, each drift does not have to increase and can even be negative. Thus, the model is more flexible than the constant drift model. Once again, the drift is a combination of the risk premium over the period and the expected rate change. The tree in Figure 4 illustrates the interest rate structure and effect of time-dependent drift.

Figure 4: Interest Rate Tree with Time-Dependent Drift



It is clear that if $\lambda_1 = \lambda_2$ then the Ho-Lee model reduces to Model 2. Also, it should not be surprising that λ_1 and λ_2 are estimated from observed market prices. In other words, r_0 is the observed one-period spot rate. λ_1 could then be estimated so that the model rate equals the observed two-period market rate. λ_2 could be calibrated from using r_0 and λ_1 and the observed market rate for a three-period security, and so on.

Arbitrage-Free Models

LO 13.5: Describe uses and benefits of the arbitrage-free models and assess the issue of fitting models to market prices.

Broadly speaking, there are two types of models: arbitrage-free models and equilibrium models. The key factor in choosing between these two models is based on the need to match

market prices. Arbitrage models are often used to quote the prices of securities that are illiquid or customized. For example, an arbitrage-free tree is constructed to properly price on-the-run Treasury securities (i.e., the model price must match the market price). Then, the arbitrage-free tree is used to predict off-the-run Treasury securities and is compared to market prices to determine if the bonds are properly valued. These arbitrage models are also commonly used for pricing derivatives based on observable prices of the underlying security (e.g., options on bonds).

There are two potential detractors of arbitrage-free models. First, calibrating to market prices is still subject to the suitability of the original pricing model. For example, if the parallel shift assumption is not appropriate, then a better fitting model (by adding drift) will still be faulty. Second, arbitrage models assume the underlying prices are accurate. This will not be the case if there is an external, temporary, exogenous shock (e.g., oversupply of securities from forced liquidation, which temporarily depresses market prices).

If the purpose of the model is relative analysis (i.e., comparing the value of one security to another), then using arbitrage-free models, which assume both securities are properly priced, is meaningless. Hence, for relative analysis, equilibrium models would be used rather than arbitrage-free models.

Vasicek Model

LO 13.6: Describe the process of constructing a simple and recombining tree for a short-term rate under the Vasicek Model with mean reversion.

The Vasicek model assumes a mean-reverting process for short-term interest rates. The underlying assumption is that the economy has an equilibrium level based on economic fundamentals such as long-run monetary supply, technological innovations, and similar factors. Therefore, if the short-term rate is above the long-run equilibrium value, the drift adjustment will be negative to bring the current rate closer to its mean-reverting level. Similarly, short-term rates below the long-run equilibrium will have a positive drift adjustment. Mean reversion is a reasonable assumption but clearly breaks down in periods of extremely high inflation (i.e., hyperinflation) or similar structural breaks.

The formal Vasicek model is as follows:

$$dr = k(\theta - r)dt + \sigma dw$$

where:
k = a parameter that measures the speed of reversion adjustment
θ = long-run value of the short-term rate assuming risk neutrality
r = current interest rate level

In this model, *k* measures the speed of the mean reversion adjustment; a high *k* will produce quicker (larger) adjustments than smaller values of *k*. A larger differential between the long-run and current rates will produce a larger adjustment in the current period.

Similar to the previous discussion, the drift term, λ, is a combination of the expected rate change and a risk premium. The risk neutrality assumption of the long-run value of the short-term rate allows θ to be approximated as:

$$\theta \approx r_l + \frac{\lambda}{k}$$

where:
r_l = the long-run true rate of interest

Let's consider a numerical example with a reversion adjustment parameter of 0.03, annual standard deviation of 150 basis points, a true long-term interest rate of 6%, a current interest rate of 6.2%, and annual drift of 0.36%. The long-run value of the short-term rate assuming risk neutrality is approximately:

$$\theta \approx 6\% + \frac{0.36\%}{0.03} = 18\%$$

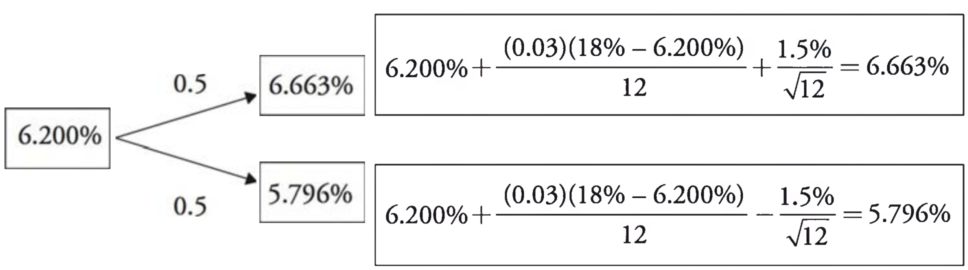
It follows that the forecasted change in the short-term rate for the next period is:

$$0.03(18\% - 6.2\%)(1/12) = 0.0295\%$$

The volatility for the monthly interval is computed as $1.5\% \times \sqrt{1/12} = 0.43\%$ (43 basis points per month).

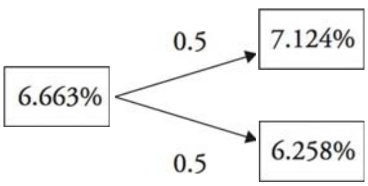
The next step is to populate the interest rate tree. Note that this tree will not recombine in the second period because the adjustment in time 2 after a downward movement in interest rates will be larger than the adjustment in time 2 following an upward movement in interest rates (since the lower node rate is further from the long-run value). This can be illustrated directly in the following calculations. Starting with r_0 = 6.2%, the interest rate tree over the first period is:

Figure 5: First Period Upper and Lower Node Calculations



If the interest rate evolves upward in the first period, we would turn to the upper node in the second period. The interest rate process can move up to 7.124% or down to 6.258%.

Figure 6: Second Period Upper Node Calculations

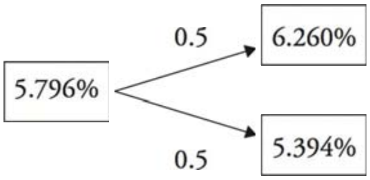


$$6.663\% + \frac{(0.03)(18\% - 6.663\%)}{12} + \frac{1.5\%}{\sqrt{12}} = 7.124\%$$

$$6.663\% + \frac{(0.03)(18\% - 6.663\%)}{12} - \frac{1.5\%}{\sqrt{12}} = 6.258\%$$

If the interest evolves downward in the first period, we would turn to the lower node in the second period. The interest rate process can move up to 6.260% or down to 5.394%.

Figure 7: Second Period Lower Node Calculations

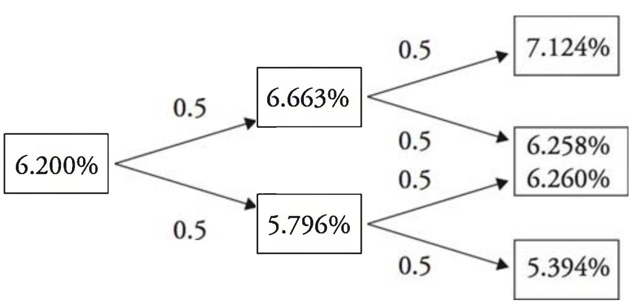


$$5.796\% + \frac{(0.03)(18\% - 5.796\%)}{12} + \frac{1.5\%}{\sqrt{12}} = 6.260\%$$

$$5.796\% + \frac{(0.03)(18\% - 5.796\%)}{12} - \frac{1.5\%}{\sqrt{12}} = 5.394\%$$

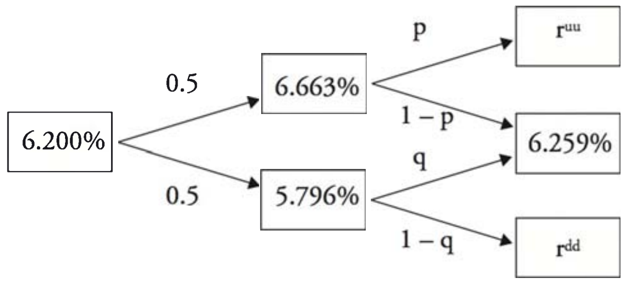
Finally, we complete the 2-period interest rate tree with mean reversion. The most interesting observation is that the model is not recombining. The up-down path leads to a 6.258% rate while the down-up path leads to a 6.260% rate. In addition, the down-up path rate is larger than the up-down path rate because the mean reversion adjustment has to be larger for the down path, as the initial interest rate was lower (5.796% versus 6.663%).

Figure 8: 2-Period Interest Rate Tree with Mean Reversion



At this point, the Vasicek model has generated a 2-period non-recombining tree of short-term interest rates. It is possible to modify the methodology so that a recombining tree is the end result. There are several ways to do this, but we will outline one straightforward method. The first step is to take an average of the two middle nodes (6.258% + 6.260%) / 2 = 6.259%. Next, we remove the assumption of 50% up and 50% down movements by generically replacing them with (p, $1 - p$) and (q, $1 - q$) as shown in Figure 9.

Figure 9: Recombining the Interest Rate Tree



The final step for recombining the tree is to solve for p and q and r^{uu} and r^{dd}. p and q are the respective probabilities of up movements in the trees in the second period after the up and down movements in the first period. r^{uu} and r^{dd} are the respective interest rates from successive (up, up and down, down) movements in the tree.

We can solve for the unknown values using a system of equations. First, we know that the average of $p \times r^{uu}$ and $(1 - p) \times 6.259\%$ must equal:

$$6.663\% + 0.03(18\% - 6.663\%)(1/12) = 6.691\%$$

Second, we can use the definition of standard deviation to equate:

$$\sqrt{p(r^{uu} - 6.691\%)^2 + (1 - p)(6.259\% - 6.691\%)^2} = 1.50\% \times \sqrt{\frac{1}{12}}$$

We would then repeat the process for the bottom portion of the tree, solving for q and r^{dd}. If the tree extends into a third period, the entire process repeats iteratively.

LO 13.7: Calculate the Vasicek Model rate change, standard deviation of the rate change, expected rate in T years, and half life.

The previous discussion encompassed the rate change in the Vasicek model and the computation of the standard deviation when solving for the parameters in the recombining tree. In this section, we turn our attention to the forecasted rate in T years.

To continue with the previous example, the current short-term rate is 6.2% with the mean-reversion parameter, k, of 0.03. The long-term mean-reverting level will eventually reach 18%, but it will take a long time since the value of k is quite small. Specifically, the current rate of 6.2% is 11.8% from its ultimate natural level and this difference will decay exponentially at the rate of mean reversion (11.8% is calculated as 18% – 6.2%). To forecast the rate in 10 years, we note that $11.8\% \times e^{(-0.03 \times 10)} = 8.74\%$. Therefore, the expected rate in 10 years is 18% – 8.74% = 9.26%.

In the Vasicek model, the expected rate in T years can be represented as the weighted average between the current short-term rate and its long-run horizon value. The weighting factor for the short-term rate decays exponentially by the speed of the mean-reverting parameter, θ:

$$r_0e^{-kT} + \theta(1 - e^{-kT})$$

A more intuitive measure for computing the forecasted rate in T years uses a factor's half-life, which measures the number of years to close half the distance between the starting rate and mean-reverting level. Numerically:

$$(18\% - 6.2\%)e^{-0.03\tau} = \tfrac{1}{2}(18\% - 6.2\%)$$

$$e^{-0.03\tau} = \tfrac{1}{2} \rightarrow \tau = \ln(2) / 0.03 = 23.1 \text{ years}$$

Professor's Note: A larger mean reversion adjustment parameter, k, *will result in a shorter half-life.*

Vasicek Model Effectiveness

LO 13.8: Describe the effectiveness of the Vasicek Model.

There are some general comments that we can make to compare mean-reverting (Vasicek) models to models without mean reversion. In development of the mean-reverting model, the parameters r_0 and θ were calibrated to match observed market prices. Hence, the mean reversion parameter not only improves the specification of the term structure, but also produces a specific term structure of volatility. Specifically, the Vasicek model will produce a term structure of volatility that is declining. Therefore, short-term volatility is overstated and long-term volatility is understated. In contrast, Model 1 with no drift generates a flat volatility of interest rates across all maturities.

Furthermore, consider an upward shift in the short-term rate. In the mean-reverting model, the short-term rate will be impacted more than long-term rates. Therefore, the Vasicek model does not imply parallel shifts from exogenous liquidity shocks. Another interpretation concerns the nature of the shock. If the shock is based on short-term economic news, then the mean reversion model implies the shock dissipates as it approaches the long-run mean. The larger the mean reversion parameter, the quicker the economic news is incorporated. Similarly, the smaller the mean reversion parameter, the longer it takes for the economic news to be assimilated into security prices. In this case, the economic news is long-lived. In contrast, shocks to short-term rates in models without drift affect all rates equally regardless of maturity (i.e., produce a parallel shift).

Key Concepts

LO 13.1

Model 1 assumes no drift and that interest rates are normally distributed. The continuously compounded instantaneous rate, r_t, will change according to:

$$dr = \sigma dw$$

Model 1 limitations:

- The no-drift assumption is not flexible enough to accommodate basic term structure shapes.
- The term structure of volatility is predicted to be flat.
- There is only one factor, the short-term rate.
- Any change in the short-term rate would lead to a parallel shift in the yield curve.

Model 2 adds a constant drift: $dr = \lambda dt + \sigma dw$. The new interest rate tree increases each node in the next time period by λdt. The drift combines the expected rate change with a risk premium. The interest rate tree is still recombining, but the middle node rate at time 2 will not equal the initial rate, as was the case with Model 1.

Model 2 limitations:

- The calibrated values of drift are often too high.
- The model requires forecasting different risk premiums for long horizons where reliable forecasts are unrealistic.

LO 13.2

The interest rate tree for Model 1 is recombining and will increase/decrease each period by the same 50% probability.

LO 13.3

The normality assumption of the terminal interest rates for Model 1 will always have a positive probability of negative interest rates. One solution to eliminate this negative rate problem is to use non-negative distributions, such as the lognormal distribution; however, this may introduce other undesirable features into the model. An alternative solution is to create an adjusted interest rate tree where negative interest rates are replaced with 0%, constraining the data from being negative.

LO 13.4

The Ho-Lee model introduces even more flexibility than Model 2 by allowing the drift term to vary from period to period (i.e., time-dependent drift). The recombined middle node at time 2 = $r_0 + (\lambda_1 + \lambda_2)dt$.

LO 13.5

Arbitrage models are often used to price securities that are illiquid or off-market (e.g., uncommon maturity for a swap). The more liquid security prices are used to develop a consistent pricing model, which in turn is used for illiquid or non-standard securities. Because arbitrage models assume the market price is "correct," the models will not be effective if there are short-term imbalances altering bond prices. Similarly, arbitrage-free models cannot be used in relative valuation analysis because the securities being compared are already assumed to be properly priced.

LO 13.6

The Vasicek model assumes mean reversion to a long-run equilibrium rate. The specific functional form of the Vasicek model is as follows:

$$dr = k(\theta - r)dt + \sigma dw$$

The parameter *k* measures the speed of the mean reversion adjustment; a high *k* will produce quicker (larger) adjustments than smaller values of *k*. Assuming there is a long-run interest rate of r_l, the long-run mean-reverting level is:

$$\theta \approx r_l + \frac{\lambda}{k}$$

The Vasicek model is not recombining. The tree can be approximated as recombining by averaging the unequal two nodes and recalibrating the associated probabilities (i.e., no longer using 50% probabilities for the up and down moves).

LO 13.7

The expected rate in *T* years can be forecasted assuming exponential decay of the difference between the current level and the mean-reverting level. The half-life, τ, can be computed as the time to move halfway between the current level and the mean-reverting level:

$$(\theta - r_0)e^{-k\tau} = \tfrac{1}{2}(\theta - r_0)$$

LO 13.8

The Vasicek model not only improves the specification of the term structure, but also produces a downward-sloping term structure of volatility. Model 1, on the other hand, predicts flat volatility of interest rates across all maturities. Model 1 implies parallel shifts from exogenous shocks while the Vasicek model does not. Long- (short-) lived economic shocks have low (high) mean reversion parameters. In contrast, in Model 1, shocks to short-term rates affect all rates equally regardless of maturity.

Concept Checkers

1. Using Model 1, assume the current short-term interest rate is 5%, annual volatility is 80bps, and dw, a normally distributed random variable with mean 0 and standard deviation $\sqrt{dt}$, has an expected value of zero. After one month, the realization of dw is –0.5. What is the change in the spot rate and the new spot rate?

	Change in Spot	New Spot Rate
A.	0.40%	5.40%
B.	–0.40%	4.60%
C.	0.80%	5.80%
D.	–0.80%	4.20%

2. Using Model 2, assume a current short-term rate of 8%, an annual drift of 50bps, and a short-term rate standard deviation of 2%. In addition, assume the ex-post realization of the dw random variable is 0.3. After constructing a 2-period interest rate tree with annual periods, what is the interest rate in the middle node at the end of year 2?
   A. 8.0%.
   B. 8.8%.
   C. 9.0%.
   D. 9.6%.

3. The Bureau of Labor Statistics has just reported an unexpected short-term increase in high-priced luxury automobiles. What is the most likely anticipated impact on a mean-reverting model of interest rates?
   A. The economic information is long-lived with a low mean-reversion parameter.
   B. The economic information is short-lived with a low mean-reversion parameter.
   C. The economic information is long-lived with a high mean-reversion parameter.
   D. The economic information is short-lived with a high mean-reversion parameter.

4. Using the Vasicek model, assume a current short-term rate of 6.2% and an annual volatility of the interest rate process of 2.5%. Also assume that the long-run mean-reverting level is 13.2% with a speed of adjustment of 0.4. Within a binomial interest rate tree, what are the upper and lower node rates after the first month?

	Upper node	Lower node
A.	6.67%	5.71%
B.	6.67%	6.24%
C.	7.16%	6.24%
D.	7.16%	5.71%

5. John Jones, FRM, is discussing the appropriate usage of mean-reverting models relative to no-drift models, models that incorporate drift, and Ho-Lee models. Jones makes the following statements:

Statement 1: Both Model 1 (no drift) and the Vasicek model assume parallel shifts from changes in the short-term rate.

Statement 2: The Vasicek model assumes decreasing volatility of future short-term rates while Model 1 assumes constant volatility of future short-term rates.

Statement 3: The constant drift model (Model 2) is a more flexible model than the Ho-Lee model.

How many of his statements are correct?
A. 0.
B. 1.
C. 2.
D. 3.

Concept Checker Answers

1. **B** Model 1 has a no-drift assumption. Using this model, the change in the interest rate is predicted as:

   $dr = \sigma dw$
   $dr = 0.8\% \times (-0.5) = -0.4\% = -40$ basis points

   Since the initial rate was 5% and $dr = -0.40\%$, the new spot rate in one month is:

   $5\% - 0.40\% = 4.60\%$

2. **C** Using Model 2 notation:

   current short-term rate, $r_0 = 8\%$
   drift, $\lambda = 0.5\%$
   standard deviation, $\sigma = 2\%$
   random variable, $dw = 0.3$
   change in time, $dt = 1$

   Since we are asked to find the interest rate at the second period middle node using Model 2, we know that the tree will recombine to the following rate: $r_0 + 2\lambda dt$.

   $8\% + 2 \times 0.5\% \times 1 = 9\%$

3. **D** The economic news is most likely short-term in nature. Therefore, the mean reversion parameter is high so the mean reversion adjustment per period will be relatively large.

4. **D** Using a Vasicek model, the upper and lower nodes for time 1 are computed as follows:

$$\text{upper node} = 6.2\% + \frac{(0.4)(13.2\% - 6.2\%)}{12} + \frac{2.5\%}{\sqrt{12}} = 7.16\%$$

$$\text{lower node} = 6.2\% + \frac{(0.4)(13.2\% - 6.2\%)}{12} - \frac{2.5\%}{\sqrt{12}} = 5.71\%$$

5. **B** Only Statement 2 is correct. The Vasicek model implies decreasing volatility and non-parallel shifts from changes in short-term rates. The Ho-Lee model is actually more general than Model 2 (the no drift and constant drift models are special cases of the Ho-Lee model).

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

The Art of Term Structure Models: Volatility and Distribution

Topic 14

Exam Focus

This topic incorporates non-constant volatility into term structure models. The generic time-dependent volatility model is very flexible and particularly useful for valuing multi-period derivatives like interest rate caps and floors. The Cox-Ingersoll-Ross (CIR) mean-reverting model suggests that the term structure of volatility increases with the level of interest rates and does not become negative. The lognormal model also has non-negative interest rates that proportionally increase with the level of the short-term rate. For the exam, you should understand how these models impact the short-term rate process, and be able to identify how a time-dependent volatility model (Model 3) differs from the models discussed in the previous topic. Also, understand the differences between the CIR and the lognormal models, as well as the differences between the lognormal models with drift and mean reversion.

Term Structure Model with Time-Dependent Volatility

LO 14.1: Describe the short-term rate process under a model with time-dependent volatility.

This topic provides a natural extension to the prior topic on modeling term structure drift by incorporating the volatility of the term structure. Following the notation convention of the previous topic, the generic continuously compounded instantaneous rate is denoted r_t and will change (over time) according to the following relationship:

$$dr = \lambda(t)dt + \sigma(t)dw$$

It is useful to note how this model augments Model 1 and the Ho-Lee model. The functional form of Model 1 (with no drift), $dr = \sigma dw$, now includes time-dependent drift and time-dependent volatility. The Ho-Lee model, $dr = \lambda(t)dt + \sigma dw$, now includes non-constant volatility. As in the earlier models, dw is normally distributed with mean 0 and standard deviation $\sqrt{dt}$.

LO 14.2: Calculate the short-term rate change and determine the behavior of the standard deviation of the rate change using a model with time dependent volatility.

The relationships between volatility in each period could take on an almost limitless number of combinations. For example, the volatility of the short-term rate in one year, $\sigma(1)$, could be 220 basis points and the volatility of the short-term rate in two years, $\sigma(2)$, could be 260 basis points. It is also entirely possible that $\sigma(1)$ could be 220 basis points and $\sigma(2)$ could be 160 basis points. To make the analysis more tractable, it is useful to assign a

specific parameterization of time-dependent volatility. Consider the following model, which is known as Model 3:

$$dr = \lambda(t)dt + \sigma e^{-\alpha t}dw$$

where:
σ = volatility at t = 0, which decreases exponentially to 0 for $\alpha > 0$

To illustrate the rate change using Model 3, assume a current short-term rate, r_0, of 5%, a drift, λ, of 0.24%, and, instead of constant volatility, include time-dependent volatility of $\sigma e^{-0.3t}$ (with initial σ = 1.50%). If we also assume the *dw* realization drawn from a normal distribution is 0.2 (with mean = 0 and standard deviation = $\sqrt{1/12}$ = 0.2887), the change in the short-term rate after one month is calculated as:

$$dr = 0.24\% \times (1/12) + 1.5\% \times e^{-0.3(1/12)} \times 0.2$$

$$dr = 0.02\% + 0.29\% = 0.31\%$$

Therefore, the expected short-term rate of 5% plus the rate change (0.31%) equals 5.31%. Note that this value would be slightly less than the value assuming constant volatility (5.32%). This difference is expected given the exponential decay in the volatility.

Model 3 Effectiveness

LO 14.3: Assess the efficacy of time-dependent volatility models.

Time-dependent volatility models add flexibility to models of future short-term rates. This is particularly useful for pricing multi-period derivatives like interest rate caps and floors. Each cap and floor is made up of single period caplets and floorlets (essentially interest rate calls and puts). The payoff to each caplet or floorlet is based on the strike rate and the current short-term rate over the next period. Hence, the pricing of the cap and floor will depend critically on the forecast of $\sigma(t)$ at several future dates.

It is impossible to describe the general behavior of the standard deviation over the relevant horizon because it will depend on the deterministic model chosen. However, there are some parallels between Model 3 and the mean-reverting drift (Vasicek) model. Specifically, if the initial volatility for both models is equal and the decay rate is the same as the mean reversion rate, then the standard deviations of the terminal distributions are exactly the same. Similarly, if the time-dependent drift in Model 3 is equal to the average interest rate path in the Vasicek model, then the two terminal distributions are identical, an even stronger observation than having the same terminal standard deviation.

There are still important differences between these models. First, Model 3 will experience a parallel shift in the yield curve from a change in the short-term rate. Second, the purpose of the model drives the choice of the model. If the model is needed to price options on fixed income instruments, then volatility dependent models are preferred to interpolate between

observed market prices. On the other hand, if the model is needed to value or hedge fixed income securities or options, then there is a rationale for choosing mean reversion models.

One criticism of time-dependent volatility models is that the market forecasts short-term volatility far out into the future, which is not likely. A compromise is to forecast volatility approaching a constant value (in Model 3, the volatility approaches 0). A point in favor of the mean reversion models is the downward-sloping volatility term structure.

Cox-Ingersoll-Ross (CIR) and Lognormal Models

LO 14.4: Describe the short-term rate process under the Cox-Ingersoll-Ross (CIR) and lognormal models.

LO 14.5: Calculate the short-term rate change and describe the basis point volatility using the CIR and lognormal models.

Another issue with the aforementioned models is that the basis-point volatility of the short-term rate is determined independently of the level of the short-term rate. This is questionable on two fronts. First, imagine a period of extremely high inflation (or even hyperinflation). The associated change in rates over the next period is likely to be larger than when rates are closer to their normal level. Second, if the economy is operating in an extremely low interest rate environment, then it seems natural that the volatility of rates will become smaller, as rates should be bounded below by zero or should be at most small, negative rates. In effect, interest rates of zero provide a downside barrier which dampens volatility.

A common solution to this problem is to apply a model where the basis-point volatility increases with the short-term rate. Whether the basis-point volatility will increase linearly or non-linearly is based on the particular functional form chosen. A popular model where the basis-point volatility (i.e., annualized volatility of *dr*) increases proportional to the square root of the rate (i.e., $\sigma\sqrt{r}$) is the **Cox-Ingersoll-Ross** (CIR) **model** where *dr* increases at a decreasing rate and σ is constant. The CIR model is shown as follows:

$$dr = k(\theta - r)dt + \sigma\sqrt{r}\,dw$$

As an illustration, let's continue with the example from LO 14.2, given the application of the CIR model. Assume a current short-term rate of 5%, a long-run value of the short-term rate, θ, of 24%, speed of the mean revision adjustment, k, of 0.04, and a volatility, σ, of 1.50%. As before, also assume the *dw* realization drawn from a normal distribution is 0.2. Using the CIR model, the change in the short-term rate after one month is calculated as:

$$dr = 0.04(24\% - 5\%)(1/12) + 1.5\%\sqrt{5\%} \times 0.2$$

$$dr = 0.063\% + 0.067\% = 0.13\%$$

Therefore, the expected short-term rate of 5% plus the rate change (0.13%) equals 5.13%.

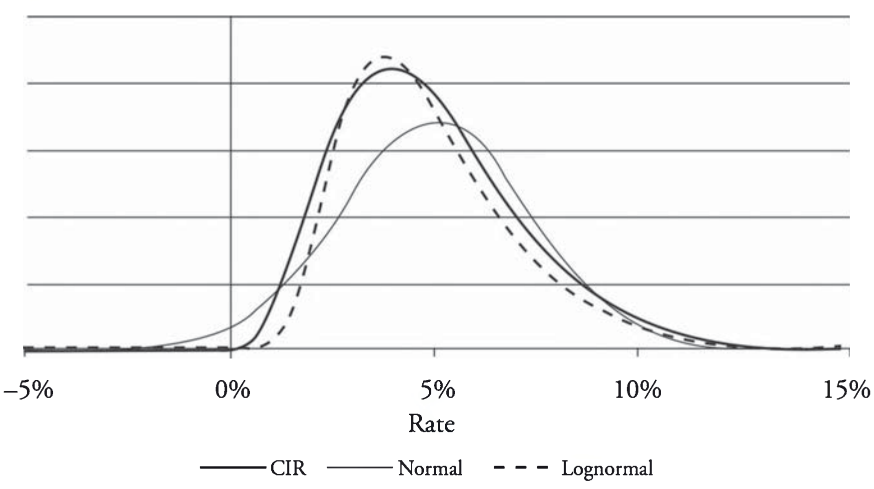
A second common specification of a model where basis-point volatility increases with the short-term rate is the **lognormal model** (Model 4). An important property of the lognormal model is that the yield volatility, σ, is constant, but basis-point volatility increases with the level of the short-term rate. Specifically, basis-point volatility is equal to σr and the functional form of the model, where σ is constant and dr increases at σr, is:

$$dr = ardt + \sigma rdw$$

For both the CIR and the lognormal models, as long as the short-term rate is not negative then a positive drift implies that the short-term rate cannot become negative. As discussed previously, this is certainly a positive feature of the models, but it actually may not be that important. For example, if a market maker feels that interest rates will be fairly flat and the possibility of negative rates would have only a marginal impact on the price, then the market maker may opt for the simpler constant volatility model rather than the more complex CIR.

The differences between the distributions of the short-term rate for the normal, CIR, and lognormal models are also important to analyze. Figure 1 compares the distributions after ten years, assuming equal means and standard deviations for all three models. As mentioned in Topic 13, the normal distribution will always imply a positive probability of negative interest rates. In addition, the longer the forecast horizon, the greater the likelihood of negative rates occurring. This can be seen directly by the left tail lying above the x-axis for rates below 0%. This is clearly a drawback to assuming a normal distribution.

Figure 1: Terminal Distributions



In contrast to the normal distribution, the lognormal and CIR terminal distributions are always non-negative and skewed right. This has important pricing implications particularly for out-of-the money options where the mass of the distributions can differ dramatically.

LO 14.6: Describe lognormal models with deterministic drift and mean reversion.

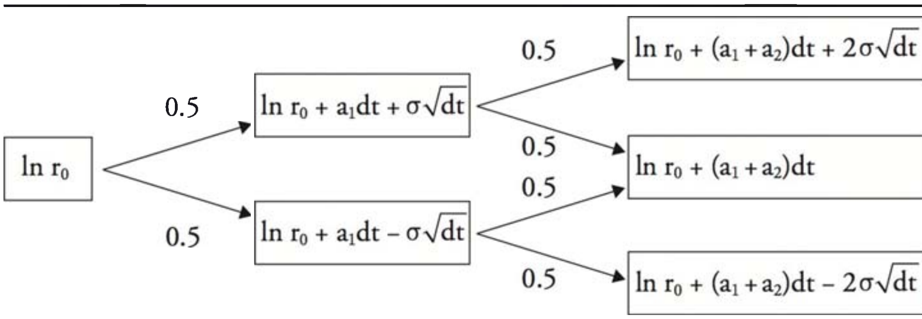
Lognormal Model with Deterministic Drift

For this LO, we detail two lognormal models, one with deterministic drift and one with mean reversion. The lognormal model with drift is shown as follows:

$$d[\ln(r)] = a(t)dt + \sigma dw$$

The natural log of the short-term rate follows a normal distribution and can be used to construct an interest rate tree based on the natural logarithm of the short-term rate. In the spirit of the Ho-Lee model, where drift can vary from period to period, the interest rate tree in Figure 2 is generated using the lognormal model with deterministic drift.

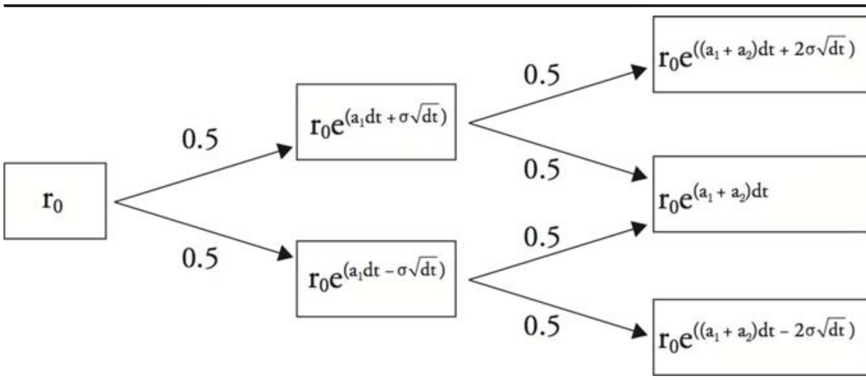
Figure 2: Interest Rate Tree with Lognormal Model (Drift)



If each node in Figure 2 is exponentiated, the tree will display the interest rates at each node. For example, the adjusted period 1 upper node would be calculated as:

$$\exp(\ln r_0 + a_1dt + \sigma\sqrt{dt}) = r_0e^{(a_1dt+\sigma\sqrt{dt})}$$

Figure 3: Lognormal Model Rates at Each Node



In contrast to the Ho-Lee model, where the drift terms are additive, the drift terms in the lognormal model are multiplicative. The implication is that all rates in the tree will always be positive since $e^x > 0$ for all x. Furthermore, since $e^x \approx 1 + x$, and if we assume $a_1 = 0$ and

$dt = 1$, then: $r_0e^{\sigma} \approx r_0(1 + \sigma)$. Hence, volatility is a percentage of the rate. For example, if $\sigma = 20\%$, then the rate in the upper node will be 20% above the current short-term rate.

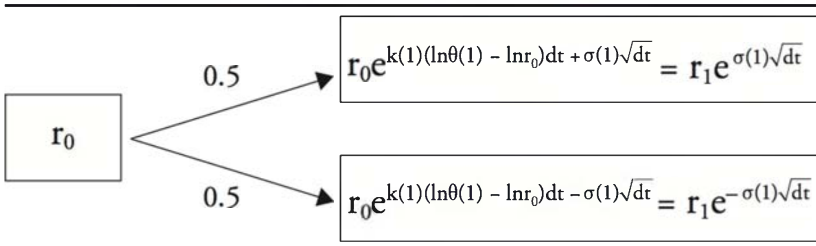
Lognormal Model with Mean Reversion

The lognormal distribution combined with a mean-reverting process is known as the Black-Karasinski model. This model is very flexible, allowing for time-varying volatility and mean reversion. In logarithmic terms, the model will appear as:

$$d[\ln(r)] = k(t)[\ln\theta(t) - \ln(r)]dt + \sigma(t)dw$$

Thus, the natural log of the short-term rate follows a normal distribution and will revert to the long-run mean of $\ln[\theta(t)]$ based on adjustment parameter $k(t)$. In addition, volatility is time-dependent, transforming the Vasicek model into a time-varying one. The interest rate tree based on this model is a bit more complicated, but it exhibits the same basic structure as previous models.

Figure 4: Interest Rate Tree with Lognormal Model (Mean Revision)



The notation r_1 is used to condense the exposition. Therefore, the ln(upper node) = $\ln r_1 + \sigma(1)\sqrt{dt}$ and ln(lower node) = $\ln r_1 - \sigma(1)\sqrt{dt}$. Following the intuition of the mean-reverting model, the tree will recombine in the second period only if:

$$k(2) = \frac{\sigma(1) - \sigma(2)}{\sigma(1)dt}$$

Recall from the previous topic that in the mean-reverting model, the nodes can be "forced" to recombine by changing the probabilities in the second period to properly value the weighted average of paths in the next period. A similar adjustment can be made in this model. However, this adjustment varies the length of time between periods (i.e., by manipulating the *dt* variable). After choosing dt_1, dt_2 is determined with the following equation:

$$k(2) = \frac{1}{dt_2}\left[1 - \frac{\sigma(2)\sqrt{dt_2}}{\sigma(1)\sqrt{dt_1}}\right]$$

Key Concepts

LO 14.1

The generic continuously compounded instantaneous rate with time-dependent drift and volatility will evolve over time according to $dr = \lambda(t)dt + \sigma(t)dw$. Special cases of this model include Model 1 ($dr = \sigma dw$) and the Ho-Lee model ($dr = \lambda(t)dt + \sigma dw$).

LO 14.2

The relationships between volatility in each period could take on an almost limitless number of combinations. To analyze this factor, it is necessary to assign a specific parameterization of time-dependent volatility such that: $dr = \lambda(t)dt + \sigma e^{-\alpha t}dw$, where σ is volatility at $t = 0$, which decreases exponentially to 0. This model is referred to as Model 3.

LO 14.3

Time-dependent volatility is very useful for pricing interest rate caps and floors that depend critically on the forecast of $\sigma(t)$ on multiple future dates. Under reasonable conditions, Model 3 and the mean-reverting drift (Vasicek) model will have the same standard deviation of the terminal distributions. One criticism of time-dependent volatility models is that the market forecasts short-term volatility far out into the future. A point in favor of the mean-reversion models is the downward-sloping volatility term structure.

LO 14.4

Two common models that avoid negative interest rates are the Cox-Ingersoll-Ross (CIR) model and lognormal model. Although avoiding negative interest rates is attractive, the non-normality of the distributions can lead to derivative mispricings.

LO 14.5

The CIR mean-reverting model has constant volatility (σ) and basis-point volatility $(\sigma\sqrt{r})$ that increases at a decreasing rate:

$$dr = k(\theta - r)dt + \sigma\sqrt{r}\,dw$$

LO 14.6

There are two lognormal models of importance: (1) lognormal with deterministic drift and (2) lognormal with mean reversion.

The lognormal model with drift is:

$$d[\ln(r)] = a(t)dt + \sigma dw$$

This model is very similar in spirit to the Ho-Lee Model with additive drift. The interest rate tree is expressed in rates, as opposed to the natural log of rates, which results in a multiplicative effect for the lognormal model with drift.

The lognormal model with mean reversion is:

$$d[\ln(r)] = k(t)[\ln\theta(t) - \ln(r)]dt + \sigma(t)dw$$

This model does not produce a naturally recombining interest rate tree. In order to force the tree to recombine, the time steps, *dt*, must be recalibrated.

Concept Checkers

1. Regarding the validity of time-dependent drift models, which of the following statements is(are) correct?

   I. Time-dependent drift models are flexible since volatility from period to period can change. However, volatility must be an increasing function of short-term rate volatilities.
   II. Time-dependent volatility functions are useful for pricing interest rate caps and floors.

   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.

2. Which of the following choices correctly characterizes basis-point volatility and yield volatility as a function of the level of the rate within the lognormal model?

	Basis-point volatility	Yield volatility
A.	increases	constant
B.	increases	decreases
C.	decreases	constant
D.	decreases	decreases

3. Which of the following statements is most likely a disadvantage of the CIR model?
   A. Interest rates are always non-negative.
   B. Option prices from the CIR distribution may differ significantly from lognormal or normal distributions.
   C. Basis-point volatility increases during periods of high inflation.
   D. Long-run interest rates hover around a mean-reverting level.

4. Which of the following statements best characterizes the differences between the Ho-Lee model with drift and the lognormal model with drift?
   A. In the Ho-Lee model and the lognormal model the drift terms are multiplicative.
   B. In the Ho-Lee model and the lognormal model the drift terms are additive.
   C. In the Ho-Lee model the drift terms are multiplicative, but in the lognormal model the drift terms are additive.
   D. In the Ho-Lee model the drift terms are additive, but in the lognormal model the drift terms are multiplicative.

5. Which of the following statements is true regarding the Black-Karasinski model?
   A. The model produces an interest rate tree that is recombining by definition.
   B. The model produces an interest rate tree that is recombining when the *dt* variable is manipulated.
   C. The model is time-varying and mean-reverting with a slow speed of adjustment.
   D. The model is time-varying and mean-reverting with a fast speed of adjustment.

Concept Checker Answers

1. **B** Time-dependent volatility models are very flexible and can incorporate increasing, decreasing, and constant short-term rate volatilities between periods. This flexibility is useful for valuing interest rate caps and floors because there is a potential payout each period, so the flexibility of changing interest rates is more appropriate than applying a constant volatility model.

2. **A** Choices B and D can be eliminated because yield volatility is constant. Basis-point volatility under the CIR model increases at a decreasing rate, whereas basis-point volatility under the lognormal model increases linearly. Therefore, basis-point volatility is an increasing function for both models.

3. **B** Choices A and C are advantages of the CIR model. Out-of-the money option prices may differ with the use of normal or lognormal distributions.

4. **D** The Ho-Lee model with drift is very flexible, allowing the drift terms each period to vary. Hence, the cumulative effect is additive. In contrast, the lognormal model with drift allows the drift terms to vary, but the cumulative effect is multiplicative.

5. **B** A feature of the time-varying, mean-reverting lognormal model is that it will not recombine naturally. Rather, the time intervals between interest rate changes are recalibrated to force the nodes to recombine. The generic model makes no prediction on the speed of the mean reversion adjustment.

The following is a review of the Market Risk Measurement and Management principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Volatility Smiles

Topic 15

Exam Focus

This topic discusses some of the reasons for the existence of volatility smiles, and how volatility affects option pricing as well as other option characteristics. Focus on the explanation of why volatility smiles exist in currency and equity options. Also, understand how volatility smiles impact the Greeks and how to interpret price jumps.

Put-Call Parity

LO 15.2: Explain the implications of put-call parity on the implied volatility of call and put options.

Recall that put-call parity is a no-arbitrage equilibrium relationship that relates European call and put option prices to the underlying asset's price and the present value of the option's strike price. In its simplest form, put-call parity can be represented by the following relationship:

$$c - p = S - PV(X)$$

where:
c = price of a call
p = price of a put
S = price of the underlying security
PV(X) = present value of the strike

PV(X) can be represented in continuous time by:

$$PV(X) = Xe^{-rT}$$

where:
r = risk-free rate
T = time left to expiration expressed in years

Since put-call parity is a no-arbitrage relationship, it will hold whether or not the underlying asset price distribution is lognormal, as required by the Black-Scholes-Merton (BSM) option pricing model.

If we simply rearrange put-call parity and denote subscripts for the option prices to indicate whether they are market or Black-Scholes-Merton option prices, the following two equations are generated:

$$p_{mkt} + S_0e^{-qt} = c_{mkt} + PV(X)$$

$$p_{BSM} + S_0e^{-qt} = c_{BSM} + PV(X)$$

Subtracting the second equation from the first gives us:

$$p_{mkt} - p_{BSM} = c_{mkt} - c_{BSM}$$

This relationship shows that, given the same strike price and time to expiration, option market prices that deviate from those dictated by the Black-Scholes-Merton model are going to deviate in the same amount whether they are for calls or puts. Since any deviation in prices will be the same, the implication is that the implied volatility of a call and put will be equal for the same strike price and time to expiration.

Volatility Smiles

LO 15.1: Define volatility smile and volatility skew.

LO 15.3: Compare the shape of the volatility smile (or skew) to the shape of the implied distribution of the underlying asset price and to the pricing of options on the underlying asset.

Actual option prices, in conjunction with the BSM model, can be used to generate implied volatilities which may differ from historical volatilities. When option traders allow implied volatility to depend on strike price, patterns of implied volatility are generated which resemble "**volatility smiles.**" These curves display implied volatility as a function of the option's strike (or exercise) price. In this topic, we will examine volatility smiles for both currency and equity options. In the case of equity options, the volatility smile is sometimes referred to as a **volatility skew** since, as we will see in LO 15.5, the volatility pattern for equity options is essentially an inverse relationship.

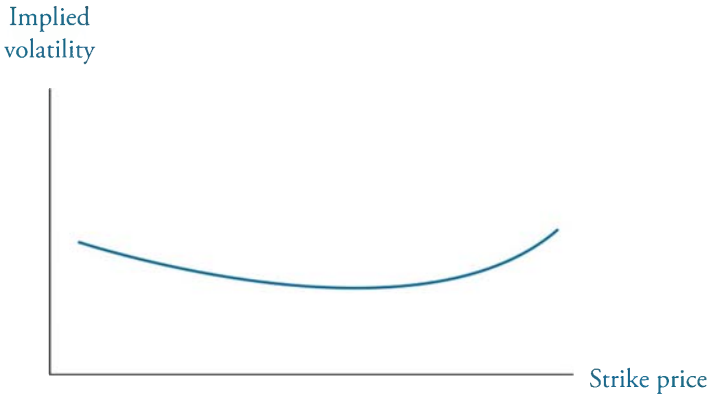
Foreign Currency Options

LO 15.4: Describe characteristics of foreign exchange rate distributions and their implications on option prices and implied volatility.

LO 15.5: Describe the volatility smile for equity options and foreign currency options and provide possible explanations for its shape.

The volatility pattern used by traders to price currency options generates implied volatilities that are higher for deep in-the-money and deep out-of-the-money options, as compared to the implied volatility for at-the-money options, as shown in Figure 1.

Figure 1: Volatility Smile for Foreign Currency Options



The easiest way to see why implied volatilities for away-from-the-money options are greater than at-the-money options is to consider the following call and put examples. For calls, a currency option is going to pay off only if the actual exchange rate is above the strike rate. For puts, on the other hand, a currency option is going to pay off only if the actual exchange rate is below the strike rate. If the implied volatilities for actual currency options are greater for away-from-the-money than at-the-money options, *currency traders must think there is a greater chance of extreme price movements than predicted by a lognormal distribution*. Empirical evidence indicates that this is the case.

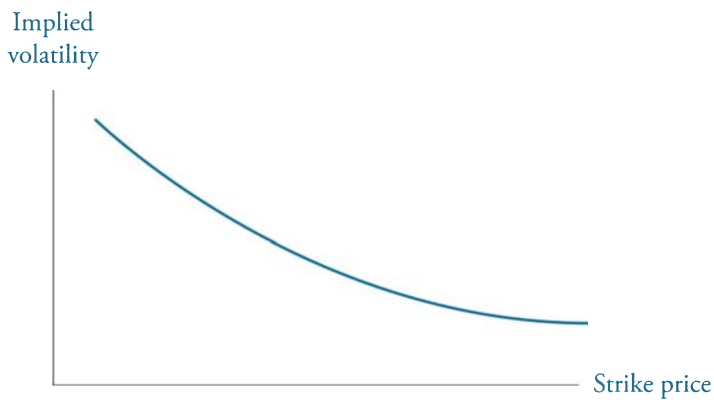
This tendency for exchange rate changes to be more extreme is a function of the fact that exchange rate volatility is not constant and frequently jumps from one level to another, which increases the likelihood of extreme currency rate levels. However, these two effects tend to be mitigated for long-dated options, which tend to exhibit less of a volatility smile pattern than shorter-dated options.

Equity Options

The equity option volatility smile is different from the currency option pattern. The smile is more of a "smirk," or skew, that shows a higher implied volatility for low strike price options (in-the-money calls and out-of-the-money puts) than for high strike price options

(in-the-money puts and out-of-the-money calls). As shown in Figure 2, there is essentially an inverse relationship between implied volatility and the strike price of equity options.

Figure 2: Volatility Smile for Equities



The volatility smirk (half-smile) exhibited by equity options translates into a left-skewed implied distribution of equity price changes. This left-skewed distribution indicates that *equity traders believe the probability of large down movements in price is greater than large up movements in price, as compared with a lognormal distribution*. Two reasons have been promoted as causing this increased implied volatility—leverage and "crashophobia."

- **Leverage.** When a firm's equity value decreases, the amount of leverage increases, which essentially increases the riskiness, or "volatility," of the underlying asset. When a firm's equity increases in value, the amount of leverage decreases, which tends to decrease the riskiness of the firm. This lowers the volatility of the underlying asset. All else held constant, there is an inverse relationship between volatility and the underlying asset's valuation.
- **Crashophobia.** The second explanation, used since the 1987 stock market crash, was coined "crashophobia" by Mark Rubinstein. Market participants are simply afraid of another market crash, so they place a premium on the probability of stock prices falling precipitously—deep out-of-the-money puts will exhibit high premiums since they provide protection against a substantial drop in equity prices. There is some support for Rubinstein's crashophobia hypothesis, because the volatility skew tends to increase when equity markets decline, but is not as noticeable when equity markets increase in value.

Alternative Methods for Studying Volatility Smiles

LO 15.6: Describe alternative ways of characterizing the volatility smile.

The volatility smiles we have characterized thus far have examined the relationship between implied volatility and strike price. Other relationships exist which allow traders to use alternative methods to study these volatility patterns. All alternatives require a replacement of the independent variable, strike price (X).

One alternative method involves replacing the strike price with strike price divided by stock price (X / S_0). This method results in a more stable volatility smile. A second alternative

approach is to substitute the strike price with strike price divided by the forward price for the underlying asset (X / F_0). The forward price would have the same maturity date as the options being assessed. Traders sometimes view the forward price as a better gauge of at the money option prices since the forward price displays the theoretical expected stock price. A third alternative method involves replacing the strike price with the option's delta. With this approach, traders are able to study volatility smiles of options other than European and American options.

Volatility Term Structure and Volatility Surfaces

LO 15.7: Describe volatility term structures and volatility surfaces and how they may be used to price options.

The **volatility term structure** is a listing of implied volatilities as a function of time to expiration for at-the-money option contracts. When short-dated volatilities are low (from historical perspectives), volatility tends to be an increasing function of maturity. When short-dated volatilities are high, volatility tends to be an inverse function of maturity. This phenomenon is related to, but has a slightly different meaning from, the mean-reverting characteristic often exhibited by implied volatility.

A **volatility surface** is nothing other than a combination of a volatility term structure with volatility smiles (i.e., those implied volatilities away-from-the-money). The surface provides guidance in pricing options with any strike or maturity structure.

A trader's primary objective is to maintain a pricing mechanism that generates option prices consistent with market pricing. Even if the implied volatility or model pricing errors change due to shifting from one pricing model to another (which could occur if traders use an alternative model to Black-Scholes-Merton), the objective is to have consistency in model-generated pricing. The volatility term structure and volatility surfaces can be used to confirm or disprove a model's accuracy and consistency in pricing.

The Option Greeks

LO 15.8: Explain the impact of the volatility smile on the calculation of the "Greeks."

Option Greeks indicate expected changes in option prices given changes in the underlying factors that affect option prices.

The problem here is that option Greeks, including delta and vega, may be affected by the implied volatility of an option. Remember these guidelines for how implied volatility may affect the Greek calculations of an option:

- The first guideline is the **sticky strike rule**, which makes an assumption that an option's implied volatility is the same over short time periods (e.g., successive days). If this is the case, the Greek calculations of an option are assumed to be unaffected, as long as the implied volatility is unchanged. If implied volatility changes, the option sensitivity calculations may not yield the correct figures.

- The second guideline is the **sticky delta rule**, which assumes the relationship between an option's price and the ratio of underlying to strike price applies in subsequent periods. The idea here is that the implied volatility reflects the moneyness of the option, so the delta calculation includes an adjustment factor for implied volatility. If the sticky delta rule holds, the option's delta will be larger than that given by the Black-Scholes-Merton formula.

Keep in mind, however, that both rules assume the volatility smile is flat for all option maturities. If this is not the case, the rules are not internally consistent and, to correct for a non-flat volatility smile, we would have to rely on an implied volatility function or tree to correctly calculate option Greeks.

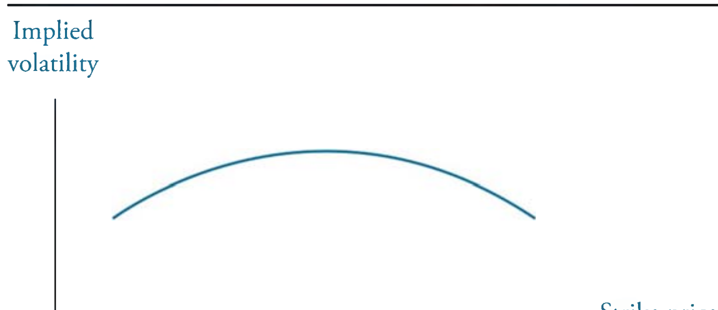
Price Jumps

LO 15.9: Explain the impact of a single asset price jump on a volatility smile.

Price jumps can occur for a number of reasons. One reason may be the expectation of a significant news event that causes the underlying asset to move either up or down by a large amount. This would cause the underlying distribution to become bimodal, but with the same expected return and standard deviation as a unimodal, or standard, price-change distribution.

Implied volatility is affected by price jumps and the probabilities assumed for either a large up or down movement. The usual result, however, is that at-the-money options tend to have a higher implied volatility than either out-of-the-money or in-the-money options. Away-from-the-money options exhibit a lower implied volatility than at-the-money options. Instead of a volatility smile, price jumps would generate a volatility frown, as in Figure 3.

Figure 3: Volatility Smile (Frown) With Price Jump



Key Concepts

LO 15.1

When option traders allow implied volatility to depend on strike price, patterns of implied volatility resemble volatility smiles.

LO 15.2

Put-call parity indicates that the deviation between market prices and Black-Scholes-Merton prices will be equivalent for calls and puts. Hence, implied volatility will be the same for calls and puts.

LO 15.3

Currency traders believe there is a greater chance of extreme price movements than predicted by a lognormal distribution. Equity traders believe the probability of large down movements in price is greater than large up movements in price, as compared with a lognormal distribution.

LO 15.4

The volatility pattern used by traders to price currency options generates implied volatilities that are higher for deep in-the-money and deep out-of-the-money options, as compared to the implied volatility for at-the-money options.

LO 15.5

The volatility smile exhibited by equity options is more of a "smirk," with implied volatility higher for low strike prices. This has been attributed to leverage and "crashophobia" effects.

LO 15.6

Alternative methods to studying volatility patterns include: replacing strike price with strike price divided by stock price, replacing strike price with strike price divided by the forward price for the underlying asset, and replacing strike price with option delta.

LO 15.7

Volatility term structures and volatility surfaces are used by traders to judge consistency in model-generated option prices.

LO 15.8

Volatility smiles that are not flat require the use of implied volatility functions or trees to correctly calculate option Greeks.

LO 15.9

Price jumps may generate volatility "frowns" instead of smiles.

Concept Checkers

1. The market price deviations for puts and calls from Black-Scholes-Merton prices indicate:
A. equivalent put and call implied volatility.
B. equivalent put and call moneyness.
C. unequal put and call implied volatility.
D. unequal put and call moneyness.

2. An empirical distribution that exhibits a fatter right tail than that of a lognormal distribution would indicate:
A. equal implied volatilities across low and high strike prices.
B. greater implied volatilities for low strike prices.
C. greater implied volatilities for high strike prices.
D. higher implied volatilities for mid-range strike prices.

3. The "sticky strike rule" assumes that implied volatility is:
A. the same across maturities for given strike prices.
B. the same for short time periods.
C. the same across strike prices for given maturities.
D. different across strike prices for given maturities.

4. Compared to at-the-money currency options, out-of-the-money currency options exhibit which of the following volatility traits?
A. Lower implied volatility.
B. A frown.
C. A smirk.
D. Higher implied volatility.

5. Which of the following regarding equity option volatility is true?
A. There is higher implied price volatility for away-from-the-money equity options.
B. "Crashophobia" suggests actual equity volatility increases when stock prices decline.
C. Compared to the lognormal distribution, traders believe the probability of large down movements in price is similar to large up movements.
D. Increasing leverage at lower equity prices suggests increasing volatility.

Concept Checker Answers

1. **A** Put-call parity indicates that the implied volatility of a call and put will be equal for the same strike price and time to expiration.

2. **C** An empirical distribution with a fat right tail generates a higher implied volatility for higher strike prices due to the increased probability of observing high underlying asset prices. The pricing indication is that in-the-money calls and out-of-the-money puts would be "expensive."

3. **B** The sticky strike rule, when applied to calculating option sensitivity measures, assumes implied volatility is the same over short time periods.

4. **D** Away-from-the-money currency options have greater implied volatility than at-the-money options. This pattern results in a volatility smile.

5. **D** There is higher implied price volatility for low strike price equity options. "Crashophobia" is based on the idea that large price declines are more likely than assumed in Black-Scholes-Merton prices, not that volatility increases when prices decline. Compared to the lognormal distribution, traders believe the probability of large down movements in price is higher than large up movements. Increasing leverage at lower equity prices suggests increasing volatility.

Self-Test: Market Risk Measurement and Management

10 Questions: 30 Minutes

1. An analyst for Z Corporation is determining the value at risk (VaR) for the corporation's profit/loss distribution that is assumed to be normally distributed. The profit/loss distribution has an annual mean of \$5 million and a standard deviation of \$3.5 million. Using a parametric approach, what is the VaR with a 99% confidence level?
   A. \$0.775 million.
   B. \$3.155 million.
   C. \$5.775 million.
   D. \$8.155 million.

2. The Basel Committee requires backtesting of actual losses to VaR calculations. How many exceptions would need to occur in a 250-day trading period for the capital multiplier to increase from three to four?
   A. two to five.
   B. five to seven.
   C. seven to nine.
   D. ten or more.

3. The top-down approach to risk aggregation assumes that a bank's portfolio can be cleanly subdivided according to market, credit, and operational risk measures. In contrast, a bottom-up approach attempts to account for interactions among various risk factors. In order to assess which approach is more appropriate, academic studies evaluate the ratio of integrated risks to separate risks. Regarding studies of top-down and bottom-up approaches, which of the following statements is incorrect?
   A. Top-down studies suggest that risk diversification is present.
   B. Bottom-up studies sometimes calculate the ratio of integrated risks to separate risks to be less than one.
   C. Bottom-up studies suggest that risk diversification should be questioned.
   D. Top-down studies calculate the ratio of integrated risks to separate risks to be greater than one.

4. Commercial Bank Z has a \$3 million loan to company A and a \$3 million loan to company B. Companies A and B each have a 5% and 4% default probability, respectively. The default correlation between companies A and B is 0.6. What is the expected loss (EL) for the commercial bank under the worst case scenario?
   a. \$83,700.
   b. \$133,900.
   c. \$165,600.
   d. \$233,800.

5. A risk manager should always pay careful attention to the limitations and advantages of applying financial models such as the value at risk (VaR) and Black-Scholes-Merton (BSM) option pricing model. Which of the following statements regarding financial models is correct?
   a. Financial models should always be calibrated using most recent market data because it is more likely to be accurate in extrapolating trends.
   b. When applying the VaR model, empirical studies imply asset returns closely follow the normal distribution.
   c. The Black-Scholes-Merton option pricing model is a good example of the advantage of using financial models because the model eliminates all mathematical inconsistences that can occur with human judgment.
   d. A good example of a limitation of a financial model is the assumption of constant volatility when applying the Black-Scholes-Merton (BSM) option pricing model.

6. Assume that a trader wishes to set up a hedge such that he sells $100,000 of a Treasury bond and buys TIPS as a hedge. Using a historical yield regression framework, assume the DV01 on the T-bond is 0.072, the DV01 on the TIPS is 0.051, and the hedge adjustment factor (regression beta coefficient) is 1.2. What is the face value of the offsetting TIPS position needed to carry out this regression hedge?
   A. $138,462.
   B. $169,412.
   C. $268,499.
   D. $280,067.

7. A constant maturity Treasury (CMT) swap pays ($1,000,000 / 2) × ($y_{CMT}$ – 9%) every six months. There is a 70% probability of an increase in the 6-month spot rate and a 60% probability of an increase in the 1-year spot rate. The rate change in all cases is 0.50% per period, and the initial y_{CMT} is 9%. What is the value of this CMT swap?
   A. $2,325.
   B. $2,229.
   C. $2,429.
   D. $905.

8. Suppose the market expects that the current 1-year rate for zero-coupon bonds with a face value of $1 will remain at 5%, but the 1-year rate in one year will be 3%. What is the 2-year spot rate for zero-coupon bonds?
   A. 3.995%.
   B. 4.088%.
   C. 4.005%.
   D. 4.115%.

9. An analyst is modeling spot rate changes using short rate term structure models. The current short-term interest rate is 5% with a volatility of 80bps. After one month passes the realization of *dw*, a normally distributed random variable with mean 0 and standard deviation $\sqrt{dt}$, is −0.5. Assume a constant interest rate drift, λ, of 0.36%. What should the analyst compute as the new spot rate?
   A. 5.37%.
   B. 4.63%.
   C. 5.76%.
   D. 4.24%.

10. Which of the following statements is incorrect regarding volatility smiles?
   A. Currency options exhibit volatility smiles because the at-the-money options have higher implied volatility than away-from-the-money options.
   B. Volatility frowns result when jumps occur in asset prices.
   C. Equity options exhibit a volatility smirk because low strike price options have greater implied volatility.
   D. Relative to currency traders, it appears that equity traders' expectations of extreme price movements are more asymmetric.

Self-Test Answers: Market Risk Measurement and Management

1. **B** The population mean and standard deviations are unknown; therefore, the standard normal z-value of 2.33 is used for a 99% confidence level.

   VaR(1%) = –5.0 million + ($3.5 million)(2.33) = –5.0 million + 8.155 million = 3.155 million (See Topic 1)

2. **D** Ten or more backtesting violations require the institution to use a capital multiplier of four. (See Topic 3)

3. **D** Top-down studies calculate this ratio to be less than one, which suggests that risk diversification is present and ignored by the separate approach. Bottom-up studies also often calculate this ratio to be less than one; however, this research has not been conclusive, and has recently found evidence of risk compounding, which produces a ratio greater than one. Thus, bottom-up studies suggests that risk diversification should be questioned. (See Topic 5)

4. **C** The default probability of company A is 5%. Thus, the standard deviation for company A is:

   $$\sqrt{0.05(1-0.05)} = 0.2179$$

   Company B has a default probability of 4% and, therefore, will have a standard deviation of 0.1960. We can now calculate the expected loss under the worst case scenario where both companies A and B are in default. Assuming that the default correlation between A and B is 0.6, the joint probability of default is:

   $$\begin{aligned}P(AB) &= 0.6\sqrt{0.05(0.95)\times 0.04(0.96)} + 0.05\times 0.04 \\ &= 0.6\sqrt{0.001824} + 0.002 = 0.0276\end{aligned}$$

   Thus, the expected loss for the commercial bank is $165,600 (= 0.0276 × $6,000,000). (See Topic 6)

5. **D** The Black-Scholes-Merton (BSM) option pricing model assumes strike prices have a constant volatility. However, numerous empirical studies find higher volatility for out-of-the-money options and a volatility skew in equity markets. Thus, this is a good example of a limitation of financial models. The choice of time period used to calibrate the parameter inputs for the model can have a big impact on the results. Risk managers used volatility and correlation estimates from pre-crisis periods during the recent financial crisis, and this resulted in significantly underestimating the risk for financial models. All financial models should be stress tested using scenarios of extreme economic conditions. VaR models often assume asset returns have a normal distribution. However, empirical studies find higher kurtosis in return distributions. High kurtosis implies a distribution with fatter tails than the normal distribution. Thus, the normal distribution is not the best assumption for the underlying distribution. Financial models contain mathematical inconsistencies. For example, in applying the BSM option pricing model for up-and-out calls and puts and down-and-out calls and puts, there are rare cases where the inputs make the model insensitive to changes in implied volatility and option maturity. (See Topic 8)

6. **B** Defining F^R and F^N as the face amounts of the real and nominal bonds, respectively, and their corresponding DV01s as $DV01^R$ and $DV01^N$, a DV01 hedge is adjusted by the hedge adjustment factor, or beta, as follows:

$$F^R = F^N \times \left(\frac{DV01^N}{DV01^R}\right) \times \beta$$

$$F^R = 100{,}000 \times \left(\frac{0.072}{0.051}\right) \times 1.2 = 169{,}412$$

(See Topic 10)

7. **A** The payoff in each period is (\$1,000,000 / 2) × (y_{CMT} – 9%). For example, the 1-year payoff of \$5,000 in the figure below is calculated as (\$1,000,000 / 2) × (10% – 9%) = \$5,000. The other numbers in the year one cells are calculated similarly.

In six months, the payoff if interest rates increase to 9.50% is (\$1,000,000 / 2) × (9.5% – 9.0%) = \$2,500. Note that the price in this cell equals the present value of the probability weighted 1-year values plus the 6-month payoff:

$$V_{6\text{ months},U} = \frac{(\$5{,}000 \times 0.6) + (\$0 \times 0.4)}{1 + \frac{0.095}{2}} + \$2{,}500 = \$5{,}363.96$$

The other cell value in six months is calculated similarly and results in a loss of \$4,418.47.

The value of the CMT swap today is the present value of the probability weighted 6-month values:

$$V_0 = \frac{(\$5{,}363.96 \times 0.7) + (-\$4{,}418.47 \times 0.3)}{1 + \frac{0.09}{2}} = \$2{,}324.62$$

Thus the correct response is A. The other answers are incorrect because they do not correctly discount the future values or omit the 6-month payoff from the 6-month values.

(See Topic 11)

8. **A** The 2-year spot rate is computed as follows:

$$\hat{r}(2) = \sqrt[2]{(1.05)(1.03)} - 1 = 3.995\%$$

(See Topic 12)

9. **B** This short rate process has an annualized drift of 0.36%, so it requires the use of Model 2 (with constant drift). The change in the spot rate is computed as:

$dr = \lambda dt + \sigma dw$

$dr = (0.36\% / 12) + (0.8\% \times -0.5) = -0.37\% = -37$ basis points

Since the initial short-term rate was 5% and *dr* is –0.37%, the new spot rate in one month is:

5% – 0.37% = 4.63%

(See Topic 13)

10. **A** Currency options exhibit volatility smiles because the at-the-money options have lower implied volatility than away-from-the-money options.

Equity traders believe that the probability of large price decreases is greater than the probability of large price increases. Currency traders' beliefs about volatility are more symmetric as there is no large skew in the distribution of expected currency values (i.e., there is a greater chance of large price movements in either direction).

(See Topic 15)

Formulas

Topic 1

profit/loss data: $P/L_t = P_t + D_t - P_{t-1}$

arithmetic return: $r_t = \dfrac{P_t + D_t - P_{t-1}}{P_{t-1}} = \dfrac{P_t + D_t}{P_{t-1}} - 1$

geometric return: $R_t = \ln\left(\dfrac{P_t + D_t}{P_{t-1}}\right)$

delta-normal VaR: $\text{VaR}(\alpha\%) = (-\mu_r + \sigma_r \times z_\alpha) \times P_{t-1}$

lognormal VaR: $\text{VaR}(\alpha\%) = P_{t-1} \times \left(1 - e^{\mu_R - \sigma_R \times z_\alpha}\right)$

standard error of a quantile: $\text{se}(q) = \dfrac{\sqrt{p(1-p)/n}}{f(q)}$

Topic 2

age-weighted historical simulation: $w(i) = \dfrac{\lambda^{i-1}(1-\lambda)}{1-\lambda^n}$

Topic 3

model accuracy test: $z = \dfrac{x - pT}{\sqrt{p(1-p)T}}$

unconditional coverage test statistic:

$$LR_{uc} = -2\ln[(1-p)^{T-N}p^N] + 2\ln\{[1-(N/T)]^{T-N}(N/T)^N\}$$

Topic 4

$V(R_P)$ is variance of portfolio return: $V(R_P) = \beta_P^2 \times V(R_M) + \sum_{i=1}^{N} w_i^2 \times \sigma_{\varepsilon,i}^2$

General market risk: $\beta_P^2 \times V(R_M)$

Specific risk: $\sum_{i=1}^{N} w_i^2 \times \sigma_{\varepsilon,i}^2$

$$\text{Undiversified VaR} = \sum_{i=1}^{N} |x_i| \times V_i$$

$$\text{Diversified VaR} = \alpha\sqrt{x'\sum x} = \sqrt{(x \times V)'R(x \times V)}$$

Topic 6

portfolio mean return: $\mu_P = w_X\mu_X + w_Y\mu_Y$

portfolio standard deviation: $\sigma_P = \sqrt{w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + 2w_Xw_Y\,\text{cov}_{XY}}$

covariance: $\text{cov}_{XY} = \dfrac{\sum_{t=1}^{n}(X_t - \mu_X)(Y_t - \mu_Y)}{n-1}$

correlation: $\rho_{XY} = \dfrac{\text{cov}_{XY}}{\sigma_X\sigma_Y}$

realized correlation: $\rho_{\text{realized}} = \dfrac{2}{n^2 - n}\sum_{i>j}\rho_{i,j}$

correlation swap payoff: notional amount × $(\rho_{\text{realized}} - \rho_{\text{fixed}})$

joint probability of default: $P(AB) = \rho_{AB}\sqrt{PD_A(1-PD_A) \times PD_B(1-PD_B)} + PD_A \times PD_B$

Topic 7

mean reversion rate: $S_t - S_{t-1} = a(\mu - S_{t-1})$

autocorrelation: $AC(\rho_t, \rho_{t-1}) = \dfrac{\text{cov}(\rho_t, \rho_{t-1})}{\sigma(\rho_t) \times \sigma(\rho_{t-1})}$

Topic 8

correlation with expectation values: $\rho_{XY} = \dfrac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - (E(X))^2} \times \sqrt{E(Y^2) - (E(Y))^2}}$

Spearman's rank correlation: $\rho_S = 1 - \dfrac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}$

Kendall's τ: $\tau = \dfrac{n_c - n_d}{n(n-1)/2}$

Topic 12

2-year spot rate: $\hat{r}(2) = \sqrt[2]{(1+r_1)(1+r_2)} - 1$

3-year spot rate: $\hat{r}(3) = \sqrt[3]{(1+r_1)(1+r_2)(1+r_3)} - 1$

Jensen's inequality: $E\left[\frac{1}{(1+r)}\right] > \frac{1}{E[1+r]}$

Topic 13

Model 1:

$dr = \sigma dw$

where:
dr = change in interest rates over small time interval, dt
dt = small time interval (measured in years)
σ = annual basis-point volatility of rate changes
dw = normally distributed random variable with mean 0 and standard deviation $\sqrt{dt}$

Model 2: $dr = \lambda dt + \sigma dw$

Vasicek model:

$dr = k(\theta - r)dt + \sigma dw$

where:
k = a parameter that measures the speed of reversion adjustment
θ = long-run value of the short-term rate assuming risk neutrality
r = current interest rate level

long-run value of short-term rate:

$$\theta \approx r_l + \frac{\lambda}{k}$$

where:
r_l = the long-run true rate of interest

Topic 14

Model 3:

$$dr = \lambda(t)dt + \sigma e^{-\alpha t}dw$$

where:
σ = volatility at $t = 0$, which decreases exponentially to 0 for $\alpha > 0$

CIR model: $dr = k(\theta - r)dt + \sigma\sqrt{r}\,dw$

Model 4: $dr = ardt + \sigma rdw$

Topic 15

put-call parity: $c - p = S - PV(X)$

Using the Cumulative Z-Table

Probability Example

Assume that the annual earnings per share (EPS) for a large sample of firms is normally distributed with a mean of \$5.00 and a standard deviation of \$1.50. What is the approximate probability of an observed EPS value falling between \$3.00 and \$7.25?

If EPS = x = \$7.25, then $z = (x - \mu)/\sigma = (\$7.25 - \$5.00)/\$1.50 = +1.50$

If EPS = x = \$3.00, then $z = (x - \mu)/\sigma = (\$3.00 - \$5.00)/\$1.50 = -1.33$

For z-value of 1.50: Use the row headed 1.5 and the column headed 0 to find the value 0.9332. This represents the area under the curve to the left of the critical value 1.50.

For z-value of –1.33: Use the row headed 1.3 and the column headed 3 to find the value 0.9082. This represents the area under the curve to the left of the critical value +1.33. The area to the left of –1.33 is 1 – 0.9082 = 0.0918.

The area between these critical values is 0.9332 – 0.0918 = 0.8414, or 84.14%.

Hypothesis Testing – One-Tailed Test Example

A sample of a stock's returns on 36 non-consecutive days results in a mean return of 2.0%. Assume the population standard deviation is 20.0%. Can we say with 95% confidence that the mean return is greater than 0%?

H_0: $\mu \leq 0.0\%$, H_A: $\mu > 0.0\%$. The test statistic = *z*-statistic = $\dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$
= (2.0 – 0.0) / (20.0 / 6) = 0.60.

The significance level = 1.0 – 0.95 = 0.05, or 5%.

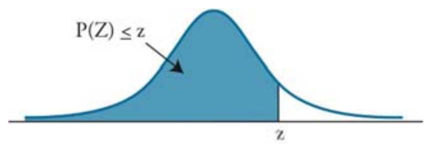
Since this is a one-tailed test with an alpha of 0.05, we need to find the value 0.95 in the cumulative *z*-table. The closest value is 0.9505, with a corresponding critical *z*-value of 1.65. Since the test statistic is less than the critical value, we fail to reject H_0.

Hypothesis Testing – Two-Tailed Test Example

Using the same assumptions as before, suppose that the analyst now wants to determine if he can say with 99% confidence that the stock's return is not equal to 0.0%.

H_0: $\mu = 0.0\%$, H_A: $\mu \neq 0.0\%$. The test statistic (*z*-value) = (2.0 – 0.0) / (20.0 / 6) = 0.60.
The significance level = 1.0 – 0.99 = 0.01, or 1%.

Since this is a two-tailed test with an alpha of 0.01, there is a 0.005 rejection region in both tails. Thus, we need to find the value 0.995 (1.0 – 0.005) in the table. The closest value is 0.9951, which corresponds to a critical *z*-value of 2.58. Since the test statistic is less than the critical value, we fail to reject H_0 and conclude that the stock's return equals 0.0%.

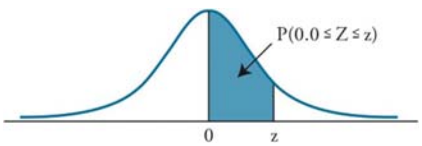


CUMULATIVE *Z*-TABLE

$P(Z \leq z) = N(z)$ for $z \geq 0$
$P(Z \leq -z) = 1 - N(z)$

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Alternative Z-Table



$P(Z \leq z) = N(z)$ for $z \geq 0$
$P(Z \leq -z) = 1 - N(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3356	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4939	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Student's T-Distribution

Level of Significance for One-Tailed Test						
df	**0.100**	**0.050**	**0.025**	**0.01**	**0.005**	**0.0005**
Level of Significance for Two-Tailed Test						
df	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.001**
1	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.599
3	1.638	2.353	3.182	4.541	5.841	12.294
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.869
6	1.440	1.943	2.447	3.143	3.707	5.959
7	1.415	1.895	2.365	2.998	3.499	5.408
8	1.397	1.860	2.306	2.896	3.355	5.041
9	1.383	1.833	2.262	2.821	3.250	4.781
10	1.372	1.812	2.228	2.764	3.169	4.587
11	1.363	1.796	2.201	2.718	3.106	4.437
12	1.356	1.782	2.179	2.681	3.055	4.318
13	1.350	1.771	2.160	2.650	3.012	4.221
14	1.345	1.761	2.145	2.624	2.977	4.140
15	1.341	1.753	2.131	2.602	2.947	4.073
16	1.337	1.746	2.120	2.583	2.921	4.015
17	1.333	1.740	2.110	2.567	2.898	3.965
18	1.330	1.734	2.101	2.552	2.878	3.922
19	1.328	1.729	2.093	2.539	2.861	3.883
20	1.325	1.725	2.086	2.528	2.845	3.850
21	1.323	1.721	2.080	2.518	2.831	3.819
22	1.321	1.717	2.074	2.508	2.819	3.792
23	1.319	1.714	2.069	2.500	2.807	3.768
24	1.318	1.711	2.064	2.492	2.797	3.745
25	1.316	1.708	2.060	2.485	2.787	3.725
26	1.315	1.706	2.056	2.479	2.779	3.707
27	1.314	1.703	2.052	2.473	2.771	3.690
28	1.313	1.701	2.048	2.467	2.763	3.674
29	1.311	1.699	2.045	2.462	2.756	3.659
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460
120	1.289	1.658	1.980	2.358	2.617	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.291

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